ED week 8 - 2 | 지난 시간 복습. electro dipole moment $\frac{dP}{d\Omega} \propto k^4 |P|^2 \sin^2\theta$

electro quadrupole moment $\frac{dP}{d\Omega} \propto k^6 |(\hat{n}\times \mathbf{Q}) \times \hat{n}$

Quadrupole moment tensor : $Q_{kl} = \int \rho(\mathbf{x}') \left[3x'_k x'_l - \cancel{x}'^2 \delta_{kl} \right] d^3\cancel{x}'$

Quadrupole moment vector = $Q_k = Q_{kl} \hat{n}_l$

example : oscillating spheroidal charge distribution.

Quadrupole moment tensor 는 이렇게 생겼다.

$$Q = \begin{pmatrix} -\frac{1}{2}Q_0 & 0 & 0 \\ 0 & -\frac{1}{2}Q_0 & 0 \\ 0 & 0 & Q_0 \end{pmatrix}$$

질문) $tr(Q) = 0$ 으로 만드는 이유 뭘까?

Quadrupole moment vector : $\mathbf{Q} = (-\frac{1}{2}Q_0, -\frac{1}{2}Q_0, Q_0)$, $|(\hat{n}\times\mathbf{Q})\times\hat{n}|^2 = |\mathbf{Q}|^2 - |\hat{n}\cdot\mathbf{Q}|^2$

$$|\mathbf{Q}|^2 = \mathbf{Q}^*\cdot\mathbf{Q} = |Q_{11}|^2 |\hat{n}_x|^2 + |Q_{22}|^2 |\hat{n}_y|^2 + |Q_{33}|^2 |\hat{n}_z|^2$$

$$= \frac{1}{4} Q_0^2 \left(\sin^2\theta\cos^2\phi + \sin^2\theta \sin^2\phi\right) + Q_0^2\cos^2\theta = \frac{1}{4}Q_0^2 \sin^2\theta + Q_0^2 \cos^2\theta$$

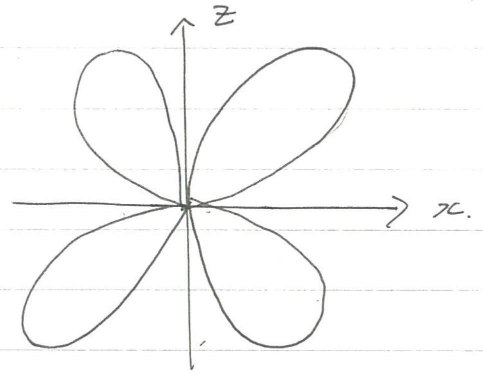
$$\hat{n}\cdot\mathbf{Q} = -\frac{1}{2}Q_0(\hat{n}_x^2 + \hat{n}_y^2) + Q_0 \hat{n}_z^2 = -\frac{1}{2}Q_0\sin^2\theta + Q_0\cos^2\theta.$$

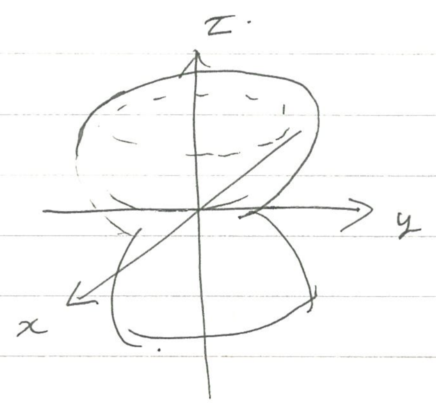
$$|\mathbf{Q}|^2 + |\hat{n}\cdot\mathbf{Q}|^2 = \frac{1}{4}Q_0^2\sin^2\theta + Q_0^2\cos^2\theta - \frac{1}{4}Q_0^2\sin^4\theta - Q_0^2\cos^4\theta + Q_0^2\sin^2\theta\cos^2\theta.$$

$$= \frac{1}{4}Q_0^2\sin^2\theta(1-\sin^2\theta) + Q_0^2\cos^2\theta(1-\cos^2\theta) + Q_0^2\sin^2\theta\cos^2\theta$$

$$= \frac{9}{4}Q_0^2\sin^2\theta\cos^2\theta.$$

따라서, $\frac{dP}{d\Omega} \propto Q_0^2 \sin^2\theta\cos^2\theta$.





Chapter 10. Scattering and Diffraction

산란을 만드는 target과 파장의 scale 차이에 따라 접근법 다름

$\lambda \gg d$: simple description in terms of lowest order induced multipoles.

$\lambda \sim d$: systematic treatment with multipole fields

$\lambda \ll d$: semi-geometrical approach.

1. Scattering at long wavelength.

A. 작은 dipole 에 의한 scattering.

입사된 전자기장에 의해, multipole이 진동한다.

induced electric/magnetic multipole. oscillating in definite phase relationship with the incident field.

이 oscillating induced multipole 자체가 radiation을 만든다.

Description via plane monochromatic wave

주변 medium의 μ와 ε는 1이라고 두자. $\hat{\varepsilon}_0$가 입사파의 polarization 방향이고, $\hat{n}_0$가 입사파의 진행 방향 unit vector 일때, 전자기장은 이렇다.

$$\mathbb{E}_{inc} = \hat{\varepsilon}_0 E_0 \exp(ik\,\hat{n}_0\cdot\mathbb{X}) \quad , \quad \mathbb{H}_{inc} = \hat{n}_0\times\mathbb{E}_{inc}/Z_0$$

그리고 산란된 전자기 파는 이렇게 나타내겠다.

$\hat{\varepsilon}$는 산란파의 편광 방향이고, $\hat{n}$은 산란파의 진행 방향 unit vector.

$\mathbb{P}$는 induced dipole moment. $\mathbb{m}$은 magnetic dipole moment

$$\mathbb{E}_{sc} = \frac{1}{4\pi\varepsilon_0} k^2 \frac{e^{ikr}}{r}\left[(\hat{n}\times\mathbb{P})\times\hat{n} - \hat{n}\times\frac{\mathbb{m}}{c}\right]$$

$$\mathbb{H}_{sc} = \hat{n}\times\mathbb{E}_{sc}/Z_0$$

Differential scattering cross-section 은 이렇게 정의된다.

$$\frac{d\sigma}{d\Omega}(\hat{n},\hat{\varepsilon};\hat{n}_0,\hat{\varepsilon}_0) = \frac{r^2\frac{1}{2Z_0}|\hat{\varepsilon}^*\cdot\mathbb{E}_{sc}|^2}{\frac{1}{2Z_0}|\hat{\varepsilon}_0^*\cdot\mathbb{E}_{inc}|^2}$$

r^2은 solid angle 때문에 그냥 붙은 것.

의미: The power radiated in the direction $\hat{n}$ with polarization $\hat{\varepsilon}$, per unit angle, per unit incident flux in the direction $\hat{n}_0$ with polarization $\hat{\varepsilon}_0$.

$$\hat{\varepsilon}^*\cdot\mathbb{E}_{sc} = \frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left[\hat{\varepsilon}^*\cdot\{(\hat{n}\times\mathbb{P})\times\hat{n}\} - \hat{\varepsilon}^*\cdot\left(\hat{n}\times\frac{\mathbb{m}}{c}\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}k^2\frac{e^{ikr}}{r}\left[\hat{\varepsilon}^*\cdot\mathbb{P} - \underbrace{(\hat{\varepsilon}^*\cdot\hat{n})}_{=0}(\hat{n}\cdot\mathbb{P}) + (\hat{n}\times\hat{\varepsilon}^*)\cdot\frac{\mathbb{m}}{c}\right]$$

(Jackson 10.4)

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left|\hat{\varepsilon}^*\cdot\mathbb{P} + (\hat{n}\times\hat{\varepsilon}^*)\cdot\frac{\mathbb{m}}{c}\right|^2}$$

Rayleigh scattering

IB. scattering by a small dielectric sphere

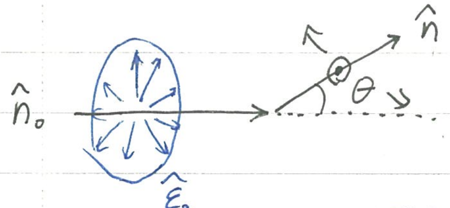
target은 dielectric constant 가 ε_r 인 작은 구.
electric dipole moment는 이렇게 형성된다.

$$\mathbb{P} = 4\pi\varepsilon_0 \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right) a^3 \mathbb{E}_{inc.} \qquad \text{(Jackson 10.5)}$$

이것을 앞선 시간에 본 $\frac{d\sigma}{d\Omega}$ 공식에 대입. magnetic dipole momen는 없다

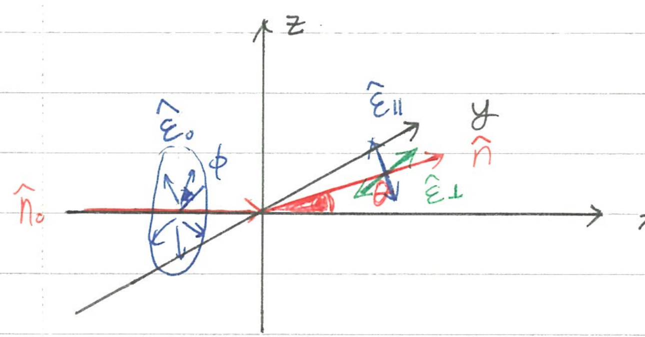
$$\frac{d\sigma}{d\Omega} = \frac{k^4}{E_0} \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 a^6 \left| \hat{\varepsilon}^* \cdot \mathbb{E}_{inc} \right|^2 = k^4 a^6 \left| \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right|^2 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 \right|^2$$

Scattering plane은 $\hat{n}$과 $\hat{n}_0$가 올라가 있는 면이다.
보통 incident wave는 편광되어 있지 않기에, $|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2$ 에서 $\hat{\varepsilon}_0$를 ~~모든~~ 모든 편광에 대해 평균내어서 계산하자.
중요한 것은 산란파의 편광이다. $\hat{\varepsilon}$가 scattering plane에 perpendicular 한지, parallel 한지에 따라 average $|\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0|^2$ 값이 달라진다.



⊙ : perpendicular component
↕ : parallel component

모든 방향의 입사 ~~평균~~에 편광에 평균내기.



plane of scattering은 x-z 평면이다
$\hat{n}_0$가 x축과 평행, $\hat{n} \cdot \hat{x} = \cos\theta$ 일때,
$\hat{\varepsilon}_0 = (0, \cos\phi, \sin\phi)$
라고 설정하면, 모든 $\hat{\varepsilon}_0$ 방향에 평균낸다는 것은 곧 ϕ 에 대해 평균 낸다는 의미다.

$\hat{\varepsilon}_{\perp} = (0, 1, 0)$

$\hat{\varepsilon}_{\parallel} = (-\sin\theta, 0, \cos\theta)$ 이므로,

$$\langle |\hat{\varepsilon}_0 \cdot \hat{\varepsilon}_{\perp}|^2 \rangle_\phi = \langle |\cos\phi|^2 \rangle_\phi = \left[\int_{-\pi}^{\pi} \cos^2\phi \, d\phi \right] \frac{1}{2\pi} = \frac{1}{2}$$

$$\langle |\hat{\varepsilon}_0 \cdot \hat{\varepsilon}_{\parallel}|^2 \rangle_\phi = \langle \sin^2\phi \cos^2\theta \rangle_\phi = \cos^2\theta \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2\phi = \frac{1}{2}\cos^2\theta.$$

각 성분에 대해 입사 편광 평균 differential scattering cross section 을 구해 보면,

$$\frac{d\sigma_{\parallel}}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \frac{\cos^2\theta}{2}, \quad \frac{d\sigma_{\perp}}{d\Omega} = k^4 a^6 \left|\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right|^2 \cdot \frac{1}{2}$$

이때, $\sigma_{\parallel}$ 는 parallel polarization 에 관한 거고, $\sigma_{\perp}$ 는 perpendicular polarization 에 관한 것이다. parallel 과 달리 perpendicular component 는 angle dependence 가 없다. $\theta = \frac{\pi}{2}$ 가 되면 산란파에서 ~~perpe~~ parallel component가 사라진다.

Scattering by a small perfectly conducting sphere

$\mathbb{P} = 4\pi\varepsilon_0 a^3 E_{inc}$ from $\Phi = -E_0\left(r - \frac{a^3}{r^2}\right)\cos\theta$ (Jackson 2.14)

from Jackson section 5.11 → $\mathbb{m} = -2\pi a^3 \mathbb{H}_{inc}$ 이 moments 들을 cross section 식에 대입

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\varepsilon_0 E_0)^2}\left| (4\pi\varepsilon_0 a^3)\underbrace{\hat{\varepsilon}^* \cdot E_{inc}}_{E_0 \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0} + (\hat{n}\times\hat{\varepsilon}^*)\cdot\underbrace{(-H_{inc})}_{\hat{n}_0 \times E_{inc}/Z_0 = \frac{E_0}{Z_0}\hat{n}_0\times\hat{\varepsilon}_0}\frac{2\pi a^3}{c}\right|^2$$

$$\frac{1}{Z_0^2} = \frac{\varepsilon_0}{\mu_0}$$

$$\frac{d\sigma}{d\Omega} = k^4 a^6 \left| \hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\hat{\varepsilon}^*)\cdot(\hat{n}_0 \times \hat{\varepsilon}_0)\right|^2$$

아까처럼, 산란파의 parallell component와 perpendicular component 각각에 대해 scattering cross section 을 구하면,

$$\frac{d\sigma_{\parallel}}{d\Omega} = \frac{k^4 a^6}{2}\left|\cos\theta - \frac{1}{2}\right|^2, \quad \frac{d\sigma_{\perp}}{d\Omega} = \frac{k^4 a^6}{2}\left|1 - \frac{1}{2}\cos\theta\right|^2$$

이걸 어찌 구했느냐 하면, 앞선 페이지에서와 똑같이, $\hat{\varepsilon}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\hat{\varepsilon}^*)\cdot(\hat{n}_0\times\hat{\varepsilon}_0)$ 을 명시적으로 계산하고, ϕ 에 대해 값을 평균낸다. 벡터들을 다시 정리하면

$\hat{\varepsilon}_0 = (0, \cos\phi, \sin\phi)$, $\hat{\varepsilon}_{\parallel} = (-\sin\theta, 0, \cos\theta)$, $\hat{\varepsilon}_{\perp} = (0, 1, 0)$

$\hat{n}_0 = (1, 0, 0)$, $\hat{n} = (\cos\theta, \sin\theta, 0)$, $\hat{n}_0 \times \hat{\varepsilon}_0 = (0, -\sin\phi, \cos\phi)$

parallel ~~perpendicular~~ component $\hat{\varepsilon}_{\parallel}$ 에 대해,

$$\hat{\varepsilon}_{\parallel}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\hat{\varepsilon}_{\parallel}^*)\cdot(\hat{n}_0\times\hat{\varepsilon}_0) = \sin\phi\cos\theta - \frac{1}{2}\sin\theta$$

$$\left\langle \left|\hat{\varepsilon}_{\parallel}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\hat{\varepsilon}_{\parallel}^*)\cdot(\hat{n}_0\times\hat{\varepsilon}_0)\right|^2 \right\rangle = \frac{1}{2}\left(\cos\theta - \frac{1}{2}\right)^2$$

perpendicular component $\hat{\varepsilon}_{\perp}$ 에 대해,

$$\hat{\varepsilon}_{\perp}^* \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\hat{\varepsilon}_{\perp}^*)\cdot(\hat{n}_0\times\hat{\varepsilon}_0) = \cos\phi - \frac{1}{2}(-\sin\theta, 0, \cos\theta)\cdot(0, -\sin\phi, \cos\phi)$$
$$= \cos\phi - \frac{1}{2}\cos\theta\cos\phi$$

$\langle |\hat{\varepsilon}^*_\perp \cdot \hat{\varepsilon}_0 - \frac{1}{2}(\hat{n}\times\hat{\varepsilon}^*_\perp)\cdot(\hat{n}_0\times\hat{\varepsilon}_0)|^2 \rangle = \frac{1}{2}\left(1-\frac{1}{2}\cos\theta\right)^2$

계산 끝!

θ 에 따른 편광도(polarization)을 정의할 수 있다.

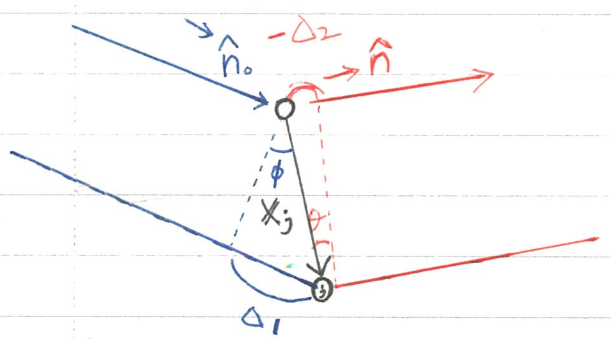
$$\Pi(\theta) = \frac{\frac{d\sigma_\perp}{d\Omega} - \frac{d\sigma_\parallel}{d\Omega}}{\frac{d\sigma_\perp}{d\Omega} + \frac{d\sigma_\parallel}{d\Omega}}$$

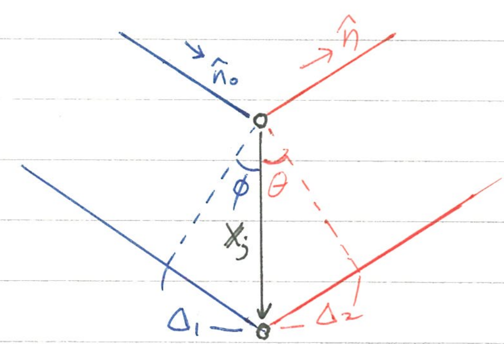
이 경우에는 $\Pi(\theta) = \dfrac{3\sin^2\theta}{5(1+\cos^2\theta) - 8\cos\theta}$

electric dipole은 $\hat{\varepsilon}^*\cdot\hat{\varepsilon}_0$ 항에서 알 수 있듯, 산란파로 편광이 직접 드러난다. 그러나 magnetization에서는 $\hat{n}\times\hat{\varepsilon}^*$ 항에서 알 수 있듯, 편광 특성이 한번 꼬아서(?) 나타나진다.

Collection of Scatterers.

일정 간격으로 떨어져 있는 여러 개의 산란자가 있다.
첫번째 입자로부터 j번째 입자로 향하는 변위 벡터를 $\mathbf{x}_j$ 라고 두자.





입사파에 대한 경로 차이는 $\Delta_1 = |\mathbf{x}_j|\sin\phi = \mathbf{x}_j\cdot\hat{n}_0$
산란파에 대한 경로 차이는 $\Delta_2 = |\mathbf{x}_j|\sin\theta = -\mathbf{x}_j\cdot\hat{n}$ 부호유의!
그래서 총 위상차는 $k(\Delta_1+\Delta_2) = k\,\mathbf{x}_j\cdot(\hat{n}_0-\hat{n})$ 만큼.
$\mathbf{q} = k(\hat{n}_0-\hat{n})$ 으로 정의하면, 위상차 $= \mathbf{q}\cdot\mathbf{x}_j$ 이다.

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi \varepsilon_0 E_0)^2} \left| \sum_j \left[\hat{\varepsilon}^* \cdot \mathbf{p}_j + (\hat{n} \times \hat{\varepsilon}^*) \cdot \frac{\mathbf{m}_j}{c} \right] e^{i\mathbf{q}\cdot\mathbf{x}_j} \right|^2$$

만약 입자들의 $\mathbf{p}$와 $\mathbf{m}$이 다 똑같다면?

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi \varepsilon_0 E_0)^2} \left(\hat{\varepsilon}^* \cdot \mathbf{p} + (\hat{n} \times \hat{\varepsilon}^*) \cdot \frac{\mathbf{m}}{c} \right) \left| \sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j} \right|^2.$$

그래서 우리는 저 summation term을 structur factor라 부른다.

$$\mathcal{F}(\mathbf{q}) = \left| \sum_j e^{i\mathbf{q}\cdot\mathbf{x}_j} \right|^2$$

1-D case

$$\mathcal{F}_{1D}(q) = \left| \sum_{n=0}^{N-1} e^{iqna} \right|^2 = \left| \sum_n^{N-1} (e^{iqa})^n \right|^2 = \left| \frac{1-e^{iqNa}}{1-e^{iqa}} \right|^2$$

$$\stackrel{?}{=} \left| \frac{\sin\left(\frac{Nqa}{2}\right)}{\sin\left(\frac{qa}{2}\right)} \right|^2.$$

왜 저게 사인이 되지?

in 3D case

$$\mathcal{F}_{3D}(\mathbf{q}) = \left| \frac{\sin(N_x q a/2)}{\sin(q a/2)} \right|^2 \left| \frac{\sin(N_Y q b/2)}{\sin(q b/2)} \right|^2 \left| \frac{\sin(N_Z q c/2)}{\sin(q c/2)} \right|^2$$

$N_x \rightarrow$ x축은 항해 몇 걸?

총 산란자 개수 : $N = N_x N_Y N_Z$.

in short wavelength case : Brag condition : $qa = 2\pi m$

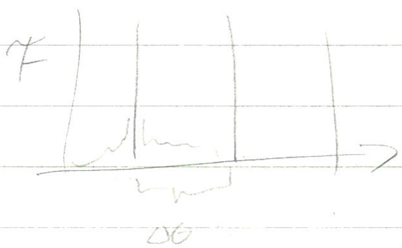
long wavelength case, $\sin\left(\frac{qa}{2}\right) \simeq \frac{qa}{2}$

$$\mathcal{F}_{long}(\mathbf{q}) = N^2 \frac{\sin^2(N_x q a/2)}{(N_x q a/2)^2} \cdot \frac{\sin^2(N_Y q a/2)}{(N_Y q a/2)^2} \cdot \frac{\sin^2(N_Z q a/2)}{(N_Z q a/2)^2}$$

$\tilde{F}(q) \propto N^2 \left(\frac{\sin X}{X}\right)^2$ $\qquad X = \frac{N_i q a_i}{2}$

$\frac{Nqa}{2} \lesssim \pi \rightarrow q = \frac{2\pi}{Na}$ $\qquad q\theta \sim \frac{2\pi}{\lambda}\theta \le \frac{2\pi}{Na}$.

$\theta \sim \frac{\Delta}{c}$



주기적인 강한 peak → 분모가 0이될

근처 작은 골짜기 → 분자가 0이될.

왜 peak 세기가 $\propto \frac{1}{N}$ 이라고 하는거지?