

ED 11th week - 1

지난 시간 Kirchhoff integral formula 를 유도함

$$\Psi(x) = \oint_S [\Psi(x') \hat{n}' \cdot \nabla' G(x, x') - G(x, x') \hat{n}' \cdot \nabla' \Psi(x')] da' \quad (\text{Jackson 10.75})$$

$$\Psi(x) = -\frac{1}{4\pi} \int_{S_1} \frac{e^{ikR}}{R} \hat{n}' \cdot \left[\nabla' \cdot \Psi + ik \left(1 + \frac{i}{kR}\right) \frac{\vec{R}}{R} \Psi(x') \right] da' \quad (\text{Jackson 10.79})$$

Kirchhoff approximation

1st S_1 에서 opening 제외하고 (그러니까, 스크린이 있는 위치에서) $\Psi = 0$, $\frac{\partial \Psi}{\partial n} = 0$

2nd the value of Ψ , $\frac{\partial \Psi}{\partial n}$ on the opening (aperture)

are equal to the values of incident waves in the absence of any screen.

→ mathematical inconsistency

만약 Ψ , $\frac{\partial \Psi}{\partial n} = 0$ 이라면, 헬름홀츠 방정식과 라플라스 방정식의 해는 모든 곳에서 $\Psi = 0$

→ remedy → 왜 이렇게 remedy 하는 건지 모르겠다.

(1) Proper choice of Green's function

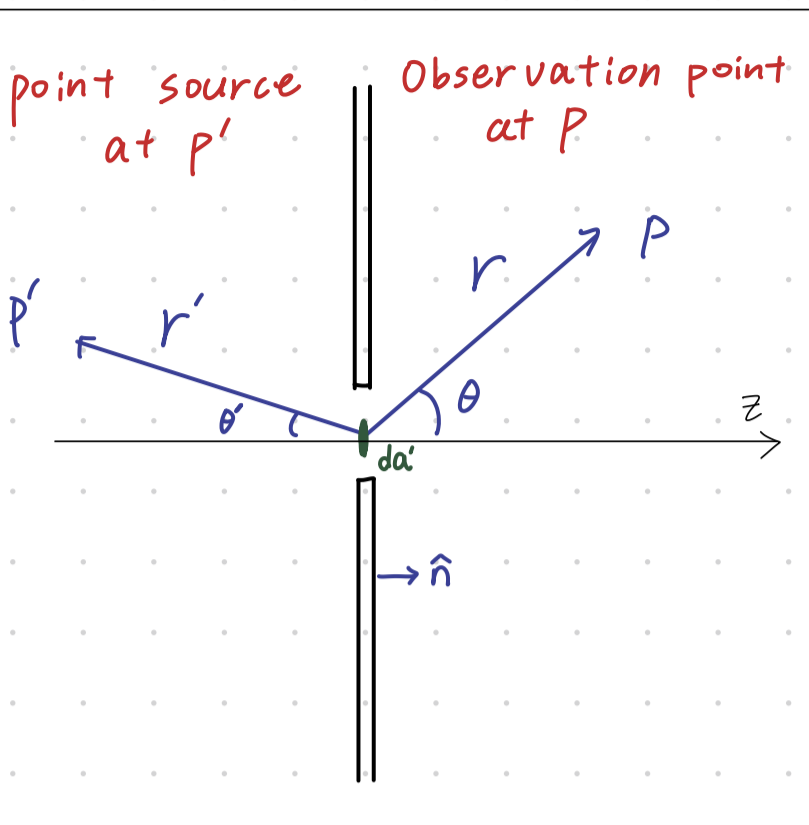
Dirichlet B.C. $G_D(x, x') = 0$ for x' on S → 이 boundary condition의 물리적 의미는?

$$\Psi(x) = \int_{S_1} \Psi(x') \frac{\partial G_D}{\partial n'} da' \quad \text{and} \quad \Psi(x) = 0 \quad \text{on} \quad S_1 \quad \text{except for opening}$$

Neumann B.C. $\frac{\partial G_N}{\partial n} = 0$ for x' on S

$$\Psi(x) = -\int_{S_1} \frac{\partial \Psi}{\partial n'}(x') G_N(x') da' \quad \text{and} \quad \frac{\partial \Psi}{\partial n'} = 0 \quad \text{on} \quad S_1 \quad \text{except for opening}$$

In finite plane screen Huygens - Fresnel principle.



$$G_{D.N}(x, x') = \frac{1}{4\pi} \left(\frac{e^{ikR}}{R} \mp \frac{e^{ikR'}}{R'} \right)$$

$$\vec{R} = x - x', \quad \vec{R}' = x - x''$$

x'' : mirror image of $x' = (x', y', -z')$

왜 이런 mirror image를 만들지? 자파의 적분 범위에서 $z'=0$ 아닌가?

$$R = |\vec{R}| = [(x-x')^2 + (y-y')^2 + (z-z')^2]^{\frac{1}{2}}$$

$$R' = |\vec{R}'| = [(x-x')^2 + (y-y')^2 + (z+z')^2]^{\frac{1}{2}}$$

$\nabla' G_{D.N}(x, x')$ 을 계산해 보자.

$$\frac{\partial}{\partial x'} R = -\frac{x'}{R}, \quad \frac{\partial}{\partial x'} = \frac{\partial R}{\partial x'} \frac{\partial}{\partial R} = -\frac{x'}{R} \frac{\partial}{\partial R}$$

$$\frac{\partial}{\partial x'} = -\frac{x'}{R} \frac{\partial}{\partial R} = -\frac{x'}{R'} \frac{\partial}{\partial R'}$$

$$\frac{\partial}{\partial y'} = -\frac{y'}{R} \frac{\partial}{\partial R} = -\frac{y'}{R'} \frac{\partial}{\partial R'}$$

$$\frac{\partial}{\partial z'} = -\frac{z'}{R} \frac{\partial}{\partial R} = +\frac{z'}{R'} \frac{\partial}{\partial R'}$$

$$\nabla' \frac{e^{ikR}}{R} = -\frac{x'}{R} \left(\frac{\partial}{\partial R} \frac{e^{ikR}}{R} \right) = \frac{x'}{R} \left(\frac{1}{R} - ik \right) \frac{e^{ikR}}{R}$$

$$\nabla' \frac{e^{ikR'}}{R'} = -\frac{x''}{R'} \left(\frac{\partial}{\partial R'} \frac{e^{ikR'}}{R'} \right) = \frac{x''}{R'} \left(\frac{1}{R'} - ik \right) \frac{e^{ikR'}}{R'}$$

$$\nabla' G_{D.N}(x, x') = \frac{1}{4\pi} \left[\frac{x'}{R} \left(\frac{1}{R} - ik \right) \frac{e^{ikR}}{R} \mp \frac{x''}{R'} \left(\frac{1}{R'} - ik \right) \frac{e^{ikR'}}{R'} \right]$$

$$= \frac{k}{4\pi i} \left[\frac{e^{ikR}}{R} \left(1 + \frac{i}{kR} \right) \frac{x'}{R} \mp \frac{e^{ikR'}}{R'} \left(1 + \frac{i}{kR'} \right) \frac{x''}{R'} \right]$$

$$= \frac{k}{4\pi i} \left[\frac{e^{ikR}}{R} \left(1 + \frac{i}{kR} \right) \frac{x - \vec{R}}{R} \mp \frac{e^{ikR'}}{R'} \left(1 + \frac{i}{kR'} \right) \frac{x - \vec{R}'}{R'} \right]$$

아니 ... $\nabla' G_{D.N}$ 구하는 건 내가 했는데, 뭘 잘못 풀었나? 어떻게 해야 아래 식이 나오지?

$$\psi(x) = \frac{k}{2\pi i} \int_{S_1} \frac{e^{ikR}}{R} \left(1 + \frac{i}{kR} \right) \frac{\hat{n}' \cdot \vec{R}}{R} \psi(x') da' \quad (\text{Jackson 10.85})$$

comparison with (Jackson 10.79)

$$\psi(\mathbf{x}) = -\frac{i}{4\pi} \int \frac{e^{ikR}}{R} \hat{n}' \cdot [\nabla' \psi + ik \left(1 + \frac{i}{kR}\right) \frac{\vec{R}}{R} \psi(\mathbf{x}')] da'$$

Jackson 10.85 에서는 피적분항 $\hat{n}' \cdot \nabla' \psi$ 이 사라지고 나머지 항이 2배가 됨을 알 수 있다.

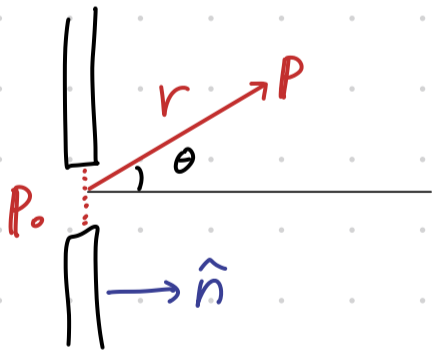
R 이 커진다면, Jackson 10.85 속 $\left(1 + \frac{i}{kR}\right)$ 에서 $\frac{i}{kR}$ 를 무시 가능.

$\frac{k}{2\pi} = \lambda$ 임을 이용하고, $\mathbf{x} = \mathbf{P}$ 이며, $\frac{\hat{n}' \cdot \vec{R}}{R} \approx \cos\theta$ 일 때,

$$\rightarrow \psi(\mathbf{P}) = \frac{1}{i\lambda} \int_{S_1} \frac{e^{ikr}}{r} \psi(\mathbf{P}_0) \cos\theta da'$$

$$\psi(\mathbf{P}) = \int \underbrace{h(\mathbf{P}, \mathbf{P}_0)}_{\text{black box}} \psi(\mathbf{P}_0) da' \quad , \quad h(\mathbf{P}, \mathbf{P}_0) = \frac{1}{i\lambda} \frac{e^{ikr}}{r} \cos\theta$$

Huygens-Fresnel equation



↑ 커린 에 뜯린 구멍들이 P0

$$\psi(\mathbf{P}) = \frac{1}{i\lambda} \int_{\Sigma} \psi(\mathbf{P}_0) \frac{e^{ikr}}{r} \cos\theta da$$

$\frac{1}{i}$ 은 phase delay 를 의미한다.

왜 구멍을 통과하면서 반파장 느려지지?

Fresnel approximation

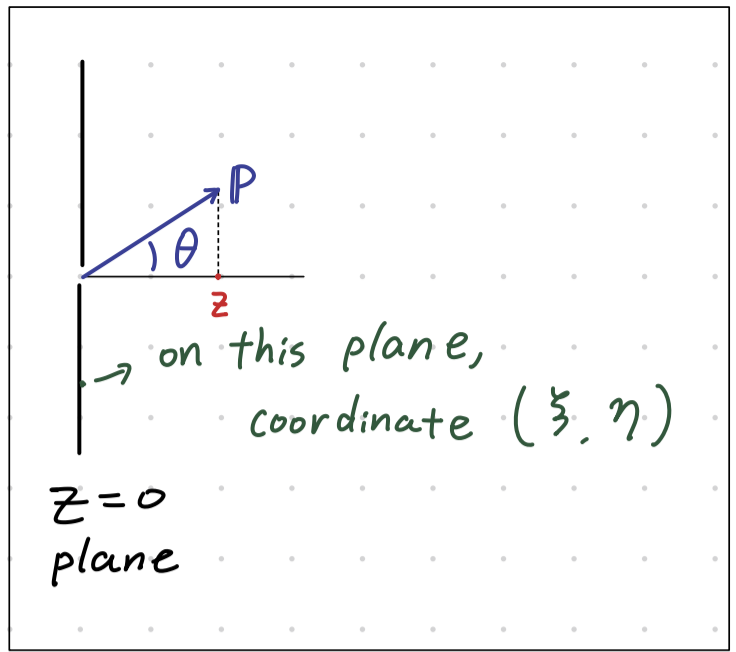
$$\cos\theta = \frac{z}{r}$$

$$\Psi(P) = \frac{1}{i\lambda} \int \Psi(P_0) \frac{e^{ikr}}{r} \cos\theta \, da$$

$$\Psi(x,y) = \frac{z}{i\lambda} \iint \Psi(\xi,\eta) \frac{\exp(ikr)}{r^2} d\xi d\eta$$

$$r = \sqrt{z^2 + (x-\xi)^2 + (y-\eta)^2}$$

$$r = z \sqrt{1 + \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2}$$



Fresnel approximation, 구멍 크기에 비해 z가 클 때,

ϵ 이 테일러 근사에 사용할 작은 수라면, $\epsilon = \left(\frac{x-\xi}{z}\right)^2 + \left(\frac{y-\eta}{z}\right)^2$

$\sqrt{1+\epsilon} \simeq 1 + \frac{1}{2}\epsilon + O(\epsilon^2)$, 따라서,

$$r \simeq z \left[1 + \frac{1}{2} \left(\frac{x-\xi}{z}\right)^2 + \frac{1}{2} \left(\frac{y-\eta}{z}\right)^2 \right]$$

$$\Psi(x,y) = \frac{e^{ikz}}{i\lambda z} \iint \Psi(\xi,\eta) \exp\left[i \frac{k}{2z} \left\{ (x-\xi)^2 + (y-\eta)^2 \right\} \right] d\xi d\eta$$

$$= \underbrace{\frac{e^{ikz}}{i\lambda z} \exp\left\{ i \frac{k}{2z} (x^2 + y^2) \right\}}_A \iint \Psi(\xi,\eta) e^{i \frac{k}{2z} (\xi^2 + \eta^2)} \underbrace{\exp\left[-i \frac{2\pi}{\lambda z} (x\xi + y\eta) \right]}_{(\xi,\eta) \rightarrow (x,y)} d\xi d\eta$$

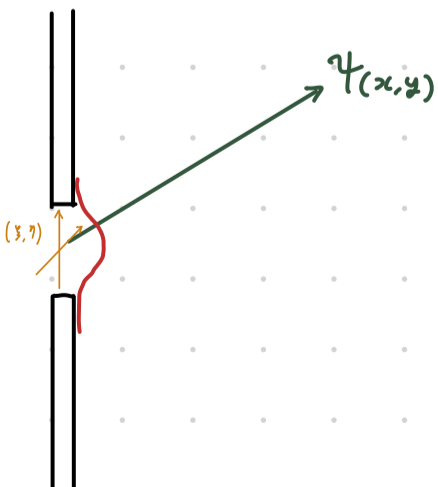
Fourier transform

$\Psi(x,y) \simeq A \times$ Fourier transform of $\left\{ \Psi(\xi,\eta) e^{i \frac{k}{2z} (\xi^2 + \eta^2)} \right\}$

incident field at aperture

Gaussian distribution

구멍 중심일 수록,
incident wave가
그대로 반영



Fraunhofer approximation

$$z \gg k(\xi^2 + \eta^2)_{\max} \rightarrow \text{to ignore } \exp\left[\frac{ik}{2z}(\xi^2 + \eta^2)\right]$$

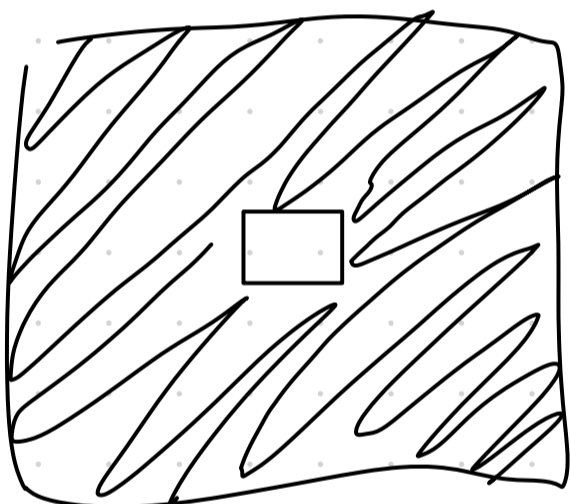
or $\lambda z > D$ when D is aperture size

$$\psi(p) = A \times \text{Fourier transform of } \psi(p_0)$$

즉, 회절 무늬는 스크린의 구멍 모양이 푸리에 변환된 것!

질문) 상수를 푸리에 변환하면 디랙델타가 나온다. 이것 지금 상황에 대입하면?

Babinet's principle.



S_1

두 complementary한 장애물

각각에 의한 회절 무늬는 똑같다.

대신 phase가 반대.

$$\begin{aligned} \psi(p) &= \frac{1}{i\lambda} \int_{S_a} \psi(p_a) \frac{e^{ikr}}{r} \cos\theta da \\ &+ \frac{1}{i\lambda} \int_{S_{a'}} \psi(p_{a'}) \frac{e^{ikr}}{r} \cos\theta da' \\ &= \int_{S_a + S_{a'}} \dots da = 0 \end{aligned}$$

$$\begin{aligned} \psi(p) &= \frac{1}{i\lambda} \int_{S_a} \psi(p_a) \frac{e^{ikr}}{r} \cos\theta da \\ &+ \frac{1}{i\lambda} \int_{S_{a'}} \psi(p_{a'}) \frac{e^{ikr}}{r} \cos\theta da' \\ &= 0 \end{aligned}$$

Inverse problem.

→ 이론상 회절 무늬를 역푸리에 변환하면

장애물의 모양을 알 수 있으나,

측정할 수 있는 것은 intensity 뿐,

phase의 정보가 없어서 IFT가 불가능하다.