ED 13th week 1 (MON)

지난 시간 배운것.

Four vector : 로렌츠 변환을 따르는 벡터

$(U_0, \vec{U})$, $U_0 = \gamma_u C$, $\vec{U} \equiv \gamma_u \vec{u} = \frac{d\vec{x}}{d\tau}$

에너지/운동량으로 이뤄진 4-vector $\left(P_0 = \frac{E}{C}, \vec{P}\right)$
$$\begin{cases} E = \gamma m_0 c^2 = \frac{m_0 c^2}{\sqrt{1-\frac{u^2}{c^2}}} \\ \vec{P} = \gamma m_0 \vec{u} \end{cases}$$

[Mathematical properties]

⟨ Usefule notations (conventions) ⟩

Four vector를 이렇게 표기: A^i , $(i=0,1,2,3)$

$$A^i = (A^0, \vec{A})$$

Squar magnitude : $(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$

Subscrip에 대한 다른 convention 도입. $A_i = (A_0, -\vec{A}) = (A^0, -\vec{A})$

$$A_0 = A^0 \;,\; A_1 = -A^1 ,\; A_2 = -A^2 ,\; A_3 = -A^3$$

time coordinate는 부호 그대로. Space coordinate는 부호 거꾸로.

$$\begin{cases} A^i : \text{contra variant} \\ A_i : \text{covariant} \end{cases}$$

Squre of 4-vectors : $\sum_{i=0}^{3} A^i A_i = (A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$

아인슈타인 합 규약을 사용하면 그냥 $A^i A_i$

비슷하게, 서로 다른 두 벡터에 대해서도, $A^i B_i = A_i B^i$

Tensar notation : A^{ik} , $A^{00} = A_{00}$, $A_{01} = -A_0^1 = -A^{01}$, $A_{11} = -A_1^1 = A^{11}$

rasing / lowering of space index changes the sign.

rasing / lowering of time index dosen't change the sign.

$$A^i_i = A^0_0 + A^1_1 + A^2_2 + A^3_3 = Tr(A)$$

$$\delta_i^k A^i = A^k$$

Metric tensor $$(g^{ik}) = (g_{ik}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$g_{ik} A^k = A_i \ , \quad g^{ik} A_k = A^i \ , \quad A^i A_i = g_{ik} A^i A^k = g^{ik} A_i A_k$$

metric tensor 를 이용해 index 위치를 바꿀 수 있다.

Partial derivative operator.

$$\frac{\partial}{\partial x'^i} = \frac{\partial x^k}{\partial x'^i} \frac{\partial}{\partial x^k}$$

$$\partial^i = \frac{\partial}{\partial x_i} = \left(\frac{\partial}{\partial x_0} , -\vec{\nabla} \right) \qquad \partial_i = \frac{\partial}{\partial x^i} = \left(\frac{\partial}{\partial x_0} , \vec{\nabla} \right)$$

Index 의 위치에 주의하라! 미분 연산은 index 위치가 뒤바뀜.

$$\partial^i A_i = \partial_i A^i = \frac{\partial A_0}{\partial x_0} + \vec{\nabla} \cdot \vec{A}$$

Laplacian in 4D

$$\Box \equiv \partial_i \partial^i = \frac{\partial^2}{\partial x^{0 2}} - \nabla^2$$

이게 바로 wave eq 에서 $\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$ 이다.

달랑베리안

Lorentz transformation

coordinate vector $X = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}$

Scalar product of two matrix $(a, b) \equiv \tilde{a} b$

Covariant coordinate vector: $gX = \begin{pmatrix} X^0 \\ -X^1 \\ -X^2 \\ -X^3 \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$

scalar product of four-vector: $\tilde{a} g b$

Linear transformation

$X' = AX$

norm (X, gX) is invariant. $\tilde{X} g X = \tilde{X}' g X'$

$$= \widetilde{AX}\, g A X$$
$$= \tilde{X} \tilde{A} g A X$$
$$\Rightarrow g = \tilde{A} g A$$

similarity transform

$\det(\tilde{A} g A) = \det g \cdot (\det A)^2 = \det g.$

$\det A = \begin{cases} 1 & \text{proper Linear transform} \\ -1 & \text{improper Linear transform} \end{cases}$

Covariance property of electrodynamics (section 11.9)

covariance of maxwell equation / Lorentz force under L.T.

$\longrightarrow$ ρ, $\vec{J}$, $\vec{E}$, $\vec{B}$ 가 LT 에 대해 잘 정의된다.

Continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \longrightarrow \partial_\alpha J^\alpha = 0$$

$J^\alpha = (c\rho, \vec{J})$ 라고 4-vector 로 정의.

4-vector of $\phi, \vec{A}$: $A^\alpha = (\phi, \vec{A})$

wave equation become : $\Box A^\alpha = \frac{4\pi}{c} J^\alpha$

Lorentz condition : $\partial_\alpha A^\alpha = 0$

갑자기 chapter 11 부터 unit 을 바꾼다. MKS unit $\rightarrow$ CGS unit (양자에서도 쓰던거)

〈두 unit 차이점 정리!〉

$\vec{E}$ and $\vec{B}$ field

맥스웰 방정식 중 포텐셜과 직결: $\vec{E} = -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} - \nabla\phi$ $\quad$ $\vec{B} = \vec{\nabla} \times \vec{A}$

$$E_x = -\frac{1}{c}\frac{\partial}{\partial t}A_x - \frac{\partial}{\partial x}\phi = -(\partial^0 A^1 - \partial^1 A^0)$$

$$B_x = \frac{\partial}{\partial z}A_y - \frac{\partial}{\partial y}A_z = -(\partial^2 A^3 - \partial^3 A^2)$$

$\vec{E}$ 와 $\vec{B}$ 전체를 anty-symmetric 2nd order tensor 로 나타낼 수 있음

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$$

Dual field strength tensor

$$\tilde{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

$F^{\alpha\beta}$ 로부터 $E \to B$, $B \to -E$ 로 바뀐 형태

맥방이 어찌 바뀌는지 보라.

$$\left\{\begin{array}{l} \nabla\cdot\vec{E} = 4\pi\rho \\ \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \end{array}\right.$$

$$\nabla\cdot\vec{E} = 4\pi\rho,\quad \nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \;\longrightarrow\; \partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c}J^\beta$$

$$\nabla\cdot\vec{B} = 0,\quad \nabla\times\vec{E} + \frac{1}{c}\frac{\partial B}{\partial t} = 0 \;\longrightarrow\; \partial_\alpha \tilde{F}^{\alpha\beta} = 0$$

활용 예시.

Transformation of electromagnetic field

전기·자기장은 텐서 $F^{\alpha\beta}$의 element.

그럼 E', B'을 어떻게 L.T로 얻을 수 있나?

E', B'는 $F'^{\alpha\beta} = \dfrac{\partial x'^{\alpha}}{\partial x^{\gamma}} \dfrac{\partial x'^{\beta}}{\partial x^{\delta}} F^{\gamma\delta}$ 의 element

A: Lorents transform matrix.

$$F' = A F \tilde{A}$$

$$F'^{\alpha\beta} = A^{\alpha\gamma} F^{\gamma\delta} A^{\delta\beta}$$

예시) x_1 방향으로 속도 $c\beta$로 이동할 때 L.T.

$$A = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$F' = \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\xi & -\sinh\xi & 0 & 0 \\ -\sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \sinh\xi & -E_x \cosh\xi & -E_y & -E_z \\ E_x \cosh\xi & -E_x \sinh\xi & -B_z & B_y \\ E_y \cosh\xi - B_z \sinh\xi & -E_y \sinh\xi + B_z \cosh\xi & 0 & -B_x \\ E_z \cosh\xi + B_y \sinh\xi & -E_z \sinh\xi - B_y \cosh\xi & B_x & 0 \end{pmatrix}$$

답: $E_x' = \gamma^2(1-\beta^2)E_x = E_x$

$E_y' = \gamma(E_y - \beta B_z)$

$E_z' = \gamma(E_z + \beta B_y)$

$B_x' = B_x$

$B_y' = \gamma(\beta E_z + B_y)$

$B_z' = \gamma(B_z - \beta E_y)$