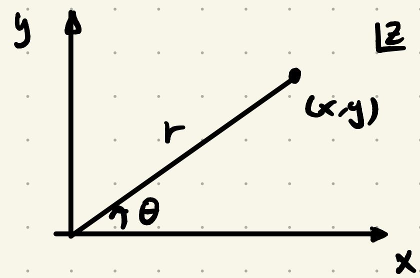
Note added. COMPLEX ANALYSIS

1. Complex variables



$\mathbb{Z}$, complex plane

'Argand diagram'

- $z = (x, y) = x + iy$

  $= r(\cos\theta + i\sin\theta) = re^{i\theta}$

- $\arg(z_1 \cdot z_2) = \arg z_1 + \arg z_2$

  $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

- De Moivre formula

$$e^{in\theta} = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Caution: Log. of complex variables : $\ln z = \ln(re^{i\theta}) = \ln r + i\theta$

However, $\theta \to \theta' = \theta + 2\pi n$ gives the same result

$$\ln z = \ln r + i(\theta + 2n\pi)$$

z, multivalued ftn.

To avoid it, $n = 0$ & $0 \le \theta < 2\pi$: principal value

" principal value of z, $\to n = 0$)

2. Complex function

$W(z) = u(x,y) + i\,v(x,y)$, u & v are pure real

e.g $W(z) = z^2 = (x+iy)^2 = (x^2 - y^2) + i2xy$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

3. Cauchy - Rieman conditions

$\Rightarrow$ Cauchy - Rieman condition

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

If dW/dz exists, Cauchy - Rieman conditions must hold.

(Proof)

- $$\lim_{\delta z \to 0} \frac{W(z+\delta z) - W(z)}{z + \delta z - z} = \lim_{\delta z \to 0} \frac{\delta W(z)}{\delta z} = \frac{dW}{dz} \text{ or } W'(z)$$

- Differentiation should be independent of the paths

$$\left.\begin{aligned} \delta z &= \delta x + i\delta y \\ \delta W &= \delta u + i\delta v \end{aligned}\right] \rightarrow \frac{\delta w}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

- path (I) $\delta y = 0$, & $\delta x \to 0$

$$\lim_{\delta z \to 0} \frac{\delta W}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i \frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

- Path (II), $\delta x = 0$ & $\delta y \to 0$

$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\delta y} + \frac{\delta v}{\delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

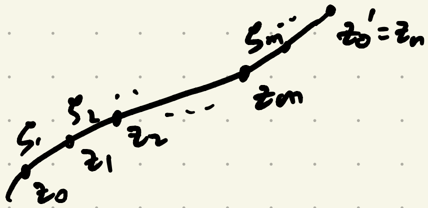
- As the $w'(z)$ should be independent of the path,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{Q.E.D.}$$

NOTE: Analytic function:

If $w(z)$ is differentiable at $z = z_0$ and nearby, then $w(z)$ is analytic at $z = z_0$

4. Cauchy integral theorem



$$S_n = \sum_{j=1}^{n} f(\xi_i)\,(z_j - z_{j-1})$$

· ξ_i, a point on the curve

· $|z_j - z_{j-1}| \to$ for $n \to \infty$

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\xi_j)\,(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)\,dz, \quad \text{Contour integral}$$

· For an analytic function, $f(z)$,

$$\boxed{\oint_c f(z)\,dz = 0}$$

$f(z)$ analytic in R
- single-valued
- partial derivatives exist
- simply connected region

Proof:

$$\oint_c f(z)\,dz = \oint (u+iv)(dx+idy)$$

$$= \oint (u\,dx - v\,dy) + i \oint (v\,dx + u\,dy) = 0$$

From, Cauchy-Rieman relation & Stokes' thm

w/ $\vec{V} = \hat{x}V_x + \hat{y}V_y \to \oint (V_x\,dx + V_y\,dy) = \int \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy$

· $u = V_x$, $v = -V_y$, gives

$$\oint (u\,dx - v\,dy) = \int \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = 0$$

Cauchy-Riemann $\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right)$

· $v = V_x$, $u = V_y$ gives

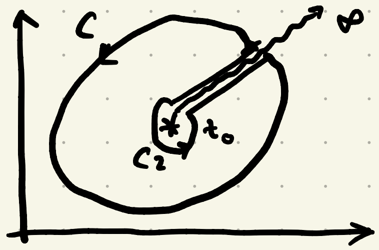
$$i \oint (v\,dx + u\,dy) = i \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0.$$

& Cauchy-Rieman

5. (4') Cauchy integral formula

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

: for $f(z)$ analytic in a closed contour 'C'



z_0, bounded by r

$$\oint_{C} \frac{f(z)}{z-z_0}\,dz - \oint_{C_2} \frac{f(z)}{z-z_0}\,dz = 0$$

$$\oint_{C_2} \frac{f(z)}{z-z_0}\,dz = \oint_{C_2} \frac{f(z_0+re^{i\theta})}{\cancel{re^{i\theta}}}\, i\cancel{re^{i\theta}}\,d\theta \to i f(z_0)\oint_{C_2} d\theta$$

$$= 2\pi i f(z_0)$$

$$\therefore \boxed{\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)} \quad \& \quad z_0 \subset C$$

Derivatives $\quad f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\displaystyle\oint \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

6. Cauchy principal value

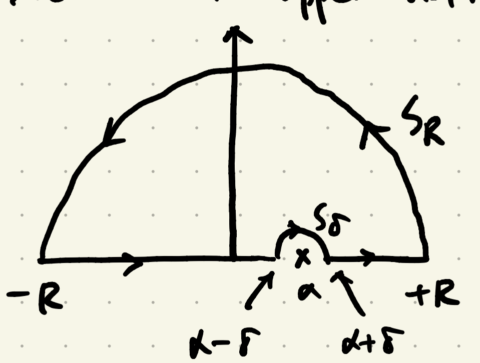
$$\boxed{P\int_{-\infty}^{\infty} \frac{f(x)}{x-\alpha}\,dx = \lim_{\delta\to 0}\left[\int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha}\,dx\right] = i\pi f(\alpha)}$$

- $f(z)$, analytic in the upper half of the complex plane

$|f(z)| \to 0$, as $|z| \to \infty$ in the upper-half plane

$$\oint \frac{f(z)}{z-\alpha}\,dz$$

for



$= 0$

$$0 = \oint_c \frac{f(z)}{z-\alpha}\,dz = \int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \int_{S_\delta} \frac{f(z)}{z-\alpha}\,dz + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha}\,dx + \int_{S_R} \frac{f(z)\,dz}{z-\alpha}$$

- For small δ,

$$P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = \int_{-R}^{\alpha-\delta} \frac{f(x)}{x-\alpha}\,dx + \int_{\alpha+\delta}^{R} \frac{f(x)}{x-\alpha}\,dx$$

Principal value integral

- $\displaystyle\int_{S_R} \frac{f(z)}{z-\alpha}\,dz$,

$z = Re^{i\theta}$, $dz = iRe^{i\theta}d\theta$, as $R\to\infty$, $f(z) \to 0$

$$\left| \int_{S_R} \frac{f(z)}{z-\alpha}\,dz \right| \le \overset{1 \text{ for } R\to\infty}{\frac{R}{R-\alpha}} \int_0^{\pi} \overset{\to 0}{|f(Re^{i\theta})|}\,d\theta = 0$$

From $|z-\alpha| = |Re^{i\theta} - \alpha| = |R^2 + \alpha^2 - 2R\alpha\cos\theta|^{1/2}$

$\ge |R^2+\alpha^2 - 2R\alpha|^{1/2} = |R-\alpha|$

$$\therefore \lim_{R\to\infty} P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = -\int_{S_\delta} \frac{f(z)}{z-\alpha}\,dz$$

$$= -f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha} - \int_{S_\delta} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz$$

- $\displaystyle -f(\alpha)\int_{S_\delta} \frac{dz}{z-\alpha} = -if(\alpha)\int_{\pi}^{0} d\theta = i\pi f(\alpha)$

$z-\alpha = \delta e^{i\theta}$

$dz = i\delta e^{i\theta}d\theta$

- if $f(z)$ is continuous at $z=\alpha$, to give

$$\int_{S_\delta} \frac{f(z)-f(\alpha)}{z-\alpha}\,dz = 0, \quad \text{Cauchy integral formula}$$

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$$\therefore \lim_{R\to\infty} P\int_{-R}^{R} \frac{f(x)}{x-\alpha}\,dx = i\pi f(\alpha)$$

or

$$P\int_{-\infty}^{\infty} \frac{f(x)}{x-\alpha}\,dx = i\pi f(\alpha)$$

For $f(x) \equiv f_R(x) + i f_I(x)$

$$\boxed{\begin{aligned} f_R(\alpha) &= \frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f_I(x)}{x-\alpha}\,dx \\ f_I(\alpha) &= -\frac{1}{\pi} P\int_{-\infty}^{\infty} \frac{f_R(x)}{x-\alpha}\,dx \end{aligned}}$$

Hilbert transformation