Maxwell equation (CGS unit)

$$\nabla \cdot \mathbb{E} = \frac{1}{\varepsilon_0}\rho \longrightarrow \text{wave}$$

$$\nabla \times \mathbb{E} = -\frac{\partial B}{\partial t} \longrightarrow \text{Potential}$$

$$\nabla \cdot \mathbb{B} = 0 \longrightarrow \text{Potential}$$

$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial \mathbb{E}}{\partial t} + \mu_0 \mathbb{J} \longrightarrow \text{wave}$$

Potentials and fields (CGS unit)

$$\mathbb{B} = \nabla \times \mathbb{A}$$

$$\mathbb{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbb{A}$$

Gauge transform → potential 을 transform

$$\mathbb{A} \longrightarrow \mathbb{A} + \nabla\Lambda$$

$$\phi \longrightarrow \phi - \frac{\partial}{\partial t}\Lambda$$

Lorentz Condition

→ potential이 맥방에 들어가면 파방이 된다.

$$\nabla \cdot \mathbb{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

Angular distribution of radiation power

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{2} Re\left[\underbrace{r^2}_{\text{면적 곱하기}} \ \underbrace{\hat{n}\cdot\left(\mathbb{E}\times\mathbb{H}^*\right)}_{\text{방사 방향의 Poynting vector}}\right]$$

Far field approximation

$\nabla \longrightarrow ik$ 로 바꿀 수 있다. 특히 $\mathbb{A}$로 부터 $\mathbb{E}$와 $\mathbb{H}$를 구할 때,

$$\mathbb{H} = \frac{1}{\mu_0}\nabla\times\mathbb{A} \longrightarrow \mathbb{H} = \frac{1}{\mu_0} ik\times\mathbb{A}$$

$$\mathbb{E} = \frac{i}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\nabla\times\mathbb{H} \longrightarrow \mathbb{E} = -\frac{1}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\, k\times\left(k\times\mathbb{A}\right)$$

Vector identity

$$\nabla\times\left(\nabla\times\mathbb{A}\right) = \text{짭플} - \text{라플} = \nabla\left(\nabla\times\mathbb{A}\right) - \nabla^2\mathbb{A}$$

4-vector notation

vector → upper index 모두 부호가 +
vector → lower index 공간성분이 부호가 −

differential operators → upper index 공간성분이 부호가 −
differential operators → lower index 모두 부호가 +

$$\partial_i A^i = \partial^i A_i = \frac{\partial A_0}{\partial X_0} + \nabla \cdot \vec{A}$$

4D - Laplacian (파동 연산자)

$$\partial_i \partial^i = \frac{\partial^2}{\partial X_0^2} - \nabla^2 \qquad \square = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2$$

---

velocity : $U^\alpha = (\gamma_u c, \gamma_u \vec{u})$

이때, $\vec{u}$는 propertime으로 구한 것.

momentum : $P^\alpha = m_0 U^\alpha = \left(\frac{E}{c}, \vec{P}\right)$

이때, $E = \gamma_u m_0 c^2$, $\vec{P} = m_0 \gamma_u \vec{u}$

current density : $J^\alpha = (c\rho, \vec{J})$
(charge density 가 함께)

potential : $A^\alpha = (\phi, \vec{A})$
scalar potential도 함께

Field strength tensor $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

example : $F^{01} = \frac{\partial A^1}{\partial x_0} + \frac{\partial A^0}{\partial x_1} = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{\partial \phi}{\partial x} = -E_x$

∂^α의 부호 유의

$$F^{12} = -\frac{\partial A^2}{\partial x_1} + \frac{\partial A^1}{\partial x_2} = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z$$

Lorentz Transform

$$\beta = \frac{v}{c} = \tanh\zeta$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\zeta \xrightarrow{\text{비상대론}} 1$$

$$\beta\gamma = \sinh\zeta$$

$$\begin{pmatrix} ct' \\ x'_{\parallel} \end{pmatrix} = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ -\sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} ct \\ x_{\parallel} \end{pmatrix}$$

$-\sin\zeta$ → $-c\sin\zeta \xrightarrow{\text{비상}} v$

$x_{\parallel}$ → 이동 방향과 평행한 성분.

Dual field strength tensor

안쪽만! 공간쪽만 부호가 바뀐다.

$$\tilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

2022 기말 문제 ① Charge in Constant E & B fields

z 방향으로 magnetic field 가, yz plane 에 electric field 가 있다. → 어떻게 해석을 해야 해?

일단 $\mathbb{E} = E_y\hat{y} + E_z\hat{z}$

Lorentz force는 $\mathbb{F} = e\mathbb{E} + \frac{e}{c}v \times \mathbb{H}$ 이라고 줌.

$\mathbb{H} = H_z\hat{z}$ 라고 두자.

(a) x, y, z 방향으로 equation of motion 을 세워라

$$v \times \mathbb{H} = \begin{pmatrix} v_y H_z \\ -v_x H_z \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = e\begin{pmatrix} 0 \\ E_y \\ E_z \end{pmatrix} + \frac{e}{c}H_z\begin{pmatrix} v_y \\ -v_x \\ 0 \end{pmatrix} = m\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix}$$

정리,
$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = \frac{e}{m}\begin{pmatrix} \frac{H_z}{c}v_y \\ E_y - \frac{H_z}{c}v_x \\ E_z \end{pmatrix}$$

(b) 속도에 대한 equation of motion 을 풀어라

제일 쉬운 건 z 성분, $v_z(t) = \frac{e}{m}E_z t + v_{z0}$

남은 건 x, y 성분의 방정식이다.

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = \frac{e}{m}\begin{pmatrix} \frac{H_z}{c}v_y \\ E_y - \frac{H_z}{c}v_x \end{pmatrix}$$

v_x 와 v_y 는 같은 진동수를 가질 것이다. v_x와 v_y 가 아래 식과 같은 형태를 가질 거라 가정.

$$\begin{cases} v_x = De^{i\omega t + i\phi} + A \\ v_y = De^{i\omega t + i\psi} + B \end{cases} \qquad \phi, \psi \in \mathbb{R}$$

방정식에 대입해서 미지수 ω, ϕ, ψ, A, B를 찾는다.

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = iD\omega e^{i\omega t}\begin{pmatrix} e^{i\phi} \\ e^{i\psi} \end{pmatrix}$$

$B = 0,\ A = \frac{c}{H_z}E_y$

$$\begin{pmatrix} \frac{H_z}{c}v_y \\ E_y - \frac{H_z}{c}v_x \end{pmatrix} = \begin{pmatrix} \frac{H_z}{c}De^{i\omega t + i\psi} + \frac{H_z}{c}B \\ E_y - \frac{H_z}{c}A - \frac{H_z}{c}e^{i\omega t + i\phi} \end{pmatrix} = D\frac{H_z}{c}e^{i\omega t}\begin{pmatrix} e^{i\psi} \\ -e^{i\phi} \end{pmatrix}$$

$\omega = \frac{eH_z}{mc}$

$$iD\omega e^{i\omega t}\begin{pmatrix} e^{i\phi} \\ e^{i\psi} \end{pmatrix} = \frac{DeH_z}{mc}e^{i\omega t}\begin{pmatrix} e^{i\psi} \\ -e^{i\phi} \end{pmatrix}, \quad \begin{pmatrix} e^{i\phi} \\ e^{i\psi} \end{pmatrix} = \frac{eH_z}{i\omega mc}\begin{pmatrix} e^{i\psi} \\ -e^{i\phi} \end{pmatrix} = \frac{eH_z}{\omega mc}\begin{pmatrix} e^{-\frac{\pi}{2}i}\cdot e^{i\psi} \\ e^{-\frac{\pi}{2}i}e^{\pi i}e^{i\phi} \end{pmatrix} = \begin{pmatrix} e^{i\psi - \frac{\pi}{2}i} \\ e^{i\phi + \frac{\pi}{2}i} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\phi} \\ e^{i\psi} \end{pmatrix} = \begin{pmatrix} e^{i\psi - \frac{\pi}{2}i} \\ e^{i\phi + \frac{\pi}{2}i} \end{pmatrix}$$ $\psi = \phi + \frac{\pi}{2}$ 임을 알 수 있다.

해를 정리하면,

$$\begin{cases} v_x = De^{i\omega t + i\phi} + \frac{c}{H_z}E_y \\ v_y = iDe^{i\omega t + i\phi} \\ v_z = \frac{e}{m}E_z t + v_{z0} \end{cases}$$

ϕ와 v_{z0}는 초기조건으로 결정된다.

(C) Frequency 와 drift velocity 에 대한 표현식을 찾아라.

방금 b번을 풀며 찾았다. Frequency는 $\omega = \frac{eH_z}{mc}$

Drift velocity는 $v_{drift} = \hat{x}\frac{c}{H_z}E_y$ 참고 : Drift velocity 의 일반식은 $\frac{c(\mathbb{E}\times\mathbb{H})}{H^2}$

(d) 운동 궤적을 그려라. initial condition : $v_y(t=0)=0$, $y(t)\geq 0$

(i) $v_x(t=0)=0$ (ii) $v_x(t=0)>0$ (iii) $v_x(t=0)<0$

공통 initial condition 에 의해, $Re(ie^{i\phi})=0$, $ie^{i\phi} = i(\cos\phi + i\sin\phi) = -\sin\phi + i\cos\phi$

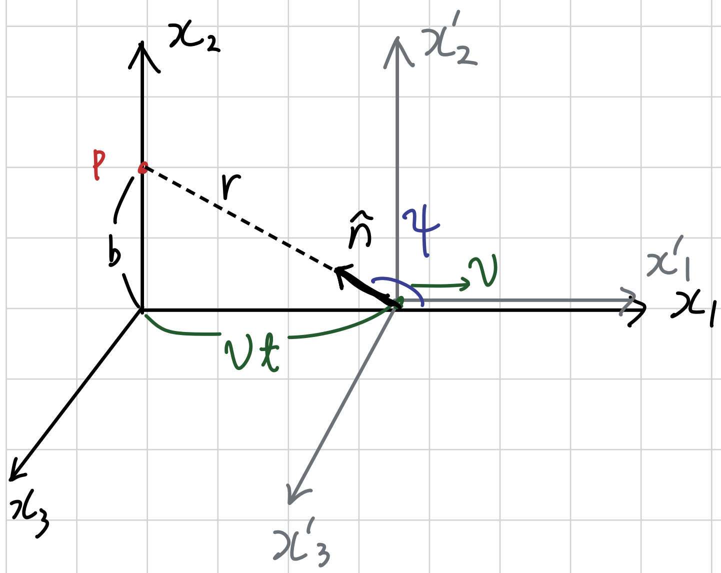
$-\sin\phi = 0$, $\phi = 0$ 으로 두자.

$\vec{v}$ 를 다시 쓰면, $v_x(t=0) = D + v_{drift}$

$$\begin{cases} v_x = Re(De^{i\omega t}) + \frac{c}{H_z}E_y = D\cos(\omega t) + v_{drift} = v_{drift}\left[\left(\frac{v_{x0}}{v_{drift}} - 1\right)\cos(\omega t) + 1\right] \\ v_y = Re(iDe^{i\omega t}) = -D\sin(\omega t) = (v_{drift} - v_{x0})\sin(\omega t) \\ v_z = \frac{e}{m}E_z t + v_{z0} \end{cases}$$

$$\begin{cases} x = v_{drift}\left[\left(\frac{v_{x0}}{v_{drift}} - 1\right)\frac{1}{\omega}\sin(\omega t) + t\right] \\ y = (v_{x0} - v_{drift})\frac{1}{\omega}\cos(\omega t) \\ z = \frac{e}{2m}E_z t^2 + v_{z0}t \end{cases}$$

2022/2023 기말 문제 ② Fields by moving charge



전하는 $\hat{x}_1$ 방향 v 속도로 이동.
관찰자는 $x_1=0$, $x_2=b$, $x_3=0$ 위치에
전하에서 관찰자로 향하는 unit vector $\hat{n}$
$\vec{v}$ 와 $\vec{n}$ 사이 각도 ψ

a) Moving frame 기준으로, 관찰자 P 위치에서 charge로 인한 field를 구하시오

풀이) Moving frame 에서는 charge가 원점 O′에 멈춰있는 정전하. 그래서 자기장은 없다.
관찰자 P는 $-v$로 이동하는 점.

P의 좌표 : $x_1' = -vt'$, $x_2' = b$, $x_3' = 0$

$\hookrightarrow$ proper time!!!

이때, t'는 moving frame의 propertime. $t' = \gamma t - \gamma\beta x_1$
P의 $x_1 = 0$ 이므로, $t' = \gamma t$ 이다.

SI unit을 이용하자.

field $\vec{E}(x_1', x_2', x_3') = q\dfrac{\hat{r}'}{(r')^2} = q\dfrac{\vec{r}'}{(r')^3}$

이때, $\vec{r}' = x_1'\hat{x}_1' + x_2'\hat{x}_2' + x_3\hat{x}_3'$, $r' = (\vec{r}'\cdot\vec{r}')^{\frac{1}{2}}$

P에서 전기장: $\vec{E}(x_1' = -vt', x_2' = b, x_3' = 0) = q\dfrac{-vt'\hat{x} + b\hat{y}}{(v^2t'^2 + b^2)^{3/2}}$
성분별로 나타내고 $t' = \gamma t$ 대입.

$$\begin{cases} E_x = \dfrac{-vq\gamma t}{[v^2\gamma^2t^2 + b^2]^{3/2}} \\[2ex] E_y = \dfrac{qb}{[v^2\gamma^2t^2 + b^2]^{3/2}} \\[2ex] E_z = B_x = B_y = B_z = 0 \end{cases}$$

b) $A^\alpha=(\phi, A)$, $F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha$ 를 이용해 field tensor 의 모든 원소를 구해라.

풀이 →

① 대각 성분은 0 이다. $F^{\alpha\alpha}=\partial^\alpha A^\alpha-\partial^\alpha A^\alpha=0$

② anti-symmetry 를 가진다. $F^{\beta\alpha}=-\left(\partial^\alpha A^\beta-\partial^\beta A^\alpha\right)=-F^{\alpha\beta}$

따라서, F^{01}, F^{02}, F^{03}, F^{12}, F^{23}, F^{31} 만 계산하면 된다.

$\partial^\alpha=\left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3}\right)$ 임을 고려. $\quad A^\alpha=(\phi, A_1, A_2, A_3)$

$$F^{0\alpha}=\partial^0 A^\alpha-\partial^\alpha A^0=\frac{1}{c}\frac{\partial \vec{A}_\alpha}{\partial t}+\frac{\partial\phi}{\partial x_\alpha}=\left[\frac{1}{c}\frac{\partial\vec{A}}{\partial t}+\nabla\phi\right]_\alpha=-\left[\vec{E}\right]_\alpha$$

$\alpha=1,2,3.$

$$F^{ij}=\partial^i A^j-\partial^j A^i=-\frac{\partial\vec{A}_j}{\partial x_i}+\frac{\partial\vec{A}_i}{\partial x_j}=-\varepsilon^{ijk}\left[\nabla\times\vec{A}\right]_k=-\left[\vec{B}\right]_k$$

$i,j=1,2,3,\ i\neq j$

즉, $F^{12}=-B_3$, $F^{23}=-B_1$, $F^{31}=-B_2$

이제 모든 것을 구했다. 정리하면,

$$F^{\alpha\beta}=\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

c) Lorentz transform matrix $A=\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ 을 이용하여, $F'=AF\tilde{A}$ 로

field 의 transformation relation 을 얻어라.

$$F\tilde{A}=\begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_z & B_y \\ E_2 & B_z & 0 & -B_x \\ E_3 & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} \gamma\beta E_1 & -\gamma E_1 & -E_2 & -E_3 \\ \gamma E_1 & -\gamma\beta E_1 & -B_z & B_y \\ \gamma E_2-\gamma\beta B_z & -\gamma\beta E_2+\gamma B_z & 0 & -B_x \\ \gamma E_3+\gamma\beta B_y & -\gamma\beta E_3-\gamma B_y & B_x & 0 \end{pmatrix}$$

$$AF\tilde{A}=\begin{pmatrix} 0 & \underbrace{(-\gamma^2+\gamma^2\beta^2)}_{-1}E_1 & -\gamma E_2+\gamma\beta B_z & -\gamma E_3-\gamma\beta B_y \\ 〃 & 0 & \gamma\beta E_2-\gamma B_z & \gamma\beta E_3+\gamma B_y \\ 〃 & 〃 & 0 & -B_x \\ 〃 & 〃 & 〃 & 0 \end{pmatrix}$$

이 $AF\tilde{A}$ 계산 결과가 $F' = \begin{pmatrix} 0 & -E_1' & -E_2' & -E_3' \\ 〃 & 0 & -B_3' & B_2' \\ 〃 & 〃 & 0 & -B_1' \\ 〃 & 〃 & 〃 & 0 \end{pmatrix}$ 과 같다고 두면,

$$\begin{cases} E_1' = E_1 \\ E_2' = \gamma(E_2 - \beta E_3) \\ E_3' = \gamma(E_3 + \beta B_2) \end{cases} \qquad \begin{cases} B_1' = B_1 \\ B_2' = \gamma(B_2 + \beta E_3) \\ B_3' = \gamma(B_3 - \beta E_2) \end{cases}$$

의 관계식이 나온다.

d) 방금 얻은 관계식을 이용해, rest frame 에서 관찰점 P 의 field 를 구하라.

위 관계식은 rest frame field (E, B) 에서 moving frame field (E', B') 로 변환, 우리는 (E', B) → (E, B) 변환식을 얻고싶고.

풀이 → 위 관계식을 그대로 쓰면 안된다. 위는 $\beta = \frac{v}{c}$ 이다.

moving frame 기준으로, rest frame 은 $-v$ 로 이동하는 frame 이다. moving frame 에서 field 가 $\vec{E}'$, $\vec{B}'$ 일 때, 그리고 rest frame 에서 field 가 $\vec{E}$, $\vec{B}$ 일 때, → 채점시 주의!

방정식을 풀거나 리리m, 역으로 생각해 restframe은 moving −cβ로 ... E,B → E',B', β → −β

$(\vec{E}', \vec{B}') \to (\vec{E}, \vec{B})$ 변환식은 위에서 얻은 것에서 $\beta \to -\beta = -\frac{v}{c}$ 를 대입한 것이다.

$$\begin{cases} E_1 = E_1' \\ E_2 = \gamma(E_2' + \beta E_3') \\ E_3 = \gamma(E_3' - \beta B_2') \end{cases} \qquad \begin{cases} B_1 = B_1' \\ B_2 = \gamma(B_2' - \beta E_3') \\ B_3 = \gamma(B_3' + \beta E_2') \end{cases}$$

그래서, E 와 B 를 구하면,

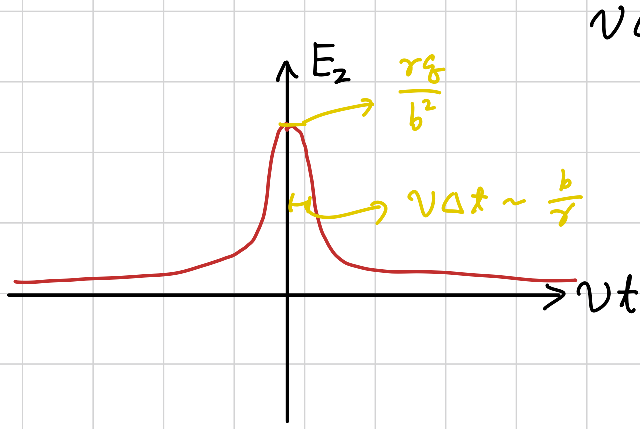
$$\begin{cases} E_1 = \dfrac{-vq\gamma t}{[(v\gamma t)^2 + b^2]^{3/2}} \\ E_2 = \dfrac{\gamma q b}{[(v\gamma t)^2 + b^2]^{3/2}} \\ E_3 = 0 \end{cases} \qquad \begin{cases} B_1 = 0 \\ B_2 = 0 \\ B_3 = \dfrac{\gamma\beta q b}{[(v\gamma t)^2 + b^2]^{3/2}} \end{cases}$$

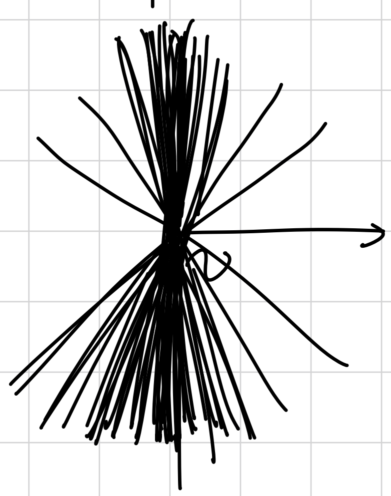
e) Fast moving charge ($\beta \sim 1$) 에 관해, E_z 의 field strength 를 plot 하고, field line 을 그려라.

γ 가 커진다.

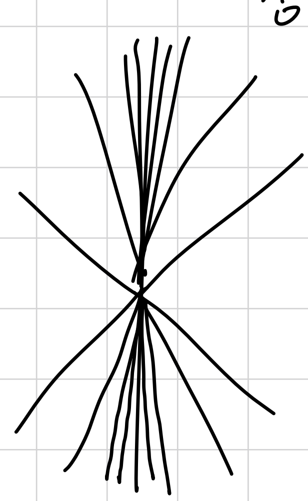
$$E_z = \frac{\gamma q b}{[(v\gamma t)^2 + b^2]^{3/2}}$$

$vt=0$ 근처에서 나타나는 peak의 폭은 $v\Delta t \propto \frac{b}{\gamma}$ 에 비례. γ 이 크면 폭이 좁아진다.





field line, 이동 방향 v와 수직한 곳으로 강하다.

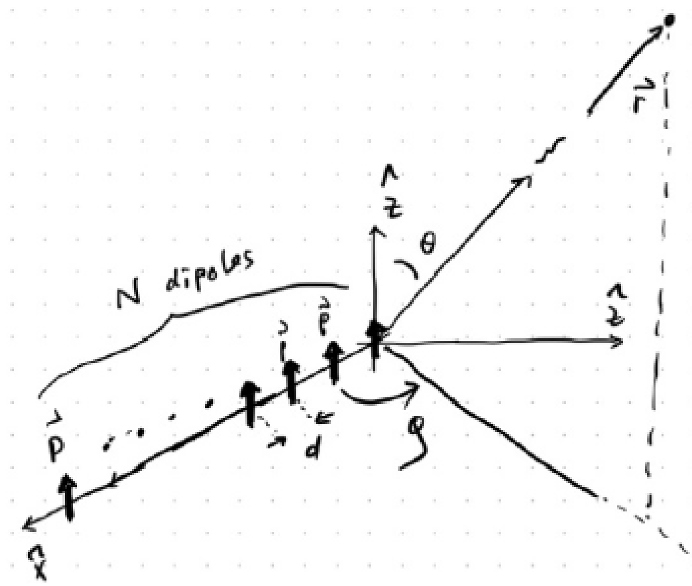


2023 기말 문제 기출 ① EM radiation from linear dipole arrays

1. **(40 points) EM radiation from linear dipole arrays.**

   Note: The vector potential at a distance *r* far from (*kr*>>1) an electric dipole moment (***p***) at the origin is obtained as $\vec{A}(\vec{r}) = -\frac{i\mu_0\omega}{4\pi}\vec{p}\frac{e^{i\vec{k}\cdot\vec{r}}}{r}$, with $\vec{k}\cdot\vec{r} = kr, here.$

   a. (**20**) Find the general expression of the vector potential for the linear array of dipole moments as shown. *The electric dipoles are displaced with the equal separation of d along the x-direction, and linear array is made of N dipole moments.* You can take $\vec{k} = k\frac{\vec{r}}{r}$, and distance from the second dipole to the observation point as $\vec{r_2} = \vec{r} + d\hat{x}$ for instance, and similarly for the other dipoles.



   b. (**10**) Plot the angular distribution of the radiation power in the xy plane. Using N=5 plot the radiation power in terms of $kd\cos\varphi$.

   c. (**10**) Find the condition for $kd\cos\varphi$ to get the maximum as well as completely suppressed radiation power for N=5.

REF: Noe that the angle dependent radiation power from a dipole moment is expressed as $\frac{dP}{d\Omega} \propto k^4|\vec{p}|^2\sin^2\theta.$

a) 풀이

문제에서 $\vec{r}_n = \vec{r} + nd\hat{x}$ 라고 했는데 이러면 틀림.

$\vec{r}_n = \vec{r} - nd\hat{x}$ 라고 해야 정확하다. $\vec{r}_n$은 n번째 dipole 로부터 관찰자 까지 벡터.

이때 원점에 있는 dipole은 0번째 이다.

$\vec{k} = k\hat{r}$ 이용

0번째 dipole로 인한 vector potential: $\vec{A}_0(\vec{r}) = -\frac{i\mu_0\omega}{4\pi}\vec{p}\frac{e^{i\vec{k}\cdot\vec{r}}}{r} = -\frac{i\mu_0\omega}{4\pi}\vec{p}\frac{e^{ikr}}{r}$

n번째 dipole로 인한 vector potential : $\vec{A}_n(\vec{r}) = -\frac{i\mu_0\omega}{4\pi}\vec{p}\frac{e^{i\vec{k}\cdot(\vec{r}_n)}}{|\vec{r}_n|}$

이때, $|\vec{r}_n| = (r^2 - 2dx + n^2d^2)^{\frac{1}{2}}$ 이지만... d가 r에 비해 작다고 치고, $|\vec{r}_n| \simeq r$ 근사.

$\vec{k}\cdot\hat{x} = k\hat{r}\cdot\hat{x} = k\sin\theta\cos\phi$

$$\vec{A}_n(\vec{r}) = -\frac{i\mu_0\omega}{4\pi}\vec{p}\,\frac{e^{-ind\vec{k}\cdot\hat{x}}\cdot e^{-i\vec{k}\cdot\vec{x}}}{r} = e^{-indk\sin\theta\cos\phi}A_0(\vec{r})$$

모든 N개 dipole 로 인한 총 vector potential은

$$\vec{A}_{net}(\vec{r}) = A_0(\vec{r})\sum_{n=0}^{N-1} e^{-indk\sin\theta\cos\phi}$$

$$= -\frac{i\mu_0\omega}{4\pi}\vec{p}\,\frac{e^{-ikx}}{r}\sum_{n=0}^{N-1} e^{-indk\sin\theta\cos\phi}$$

$\psi = dk\sin\theta\cos\phi$

$$\sum_{n=0}^{N-1} e^{-in\psi} = \frac{1-e^{-iN\psi}}{1-e^{-i\psi}}$$

---

b) 풀이.

xy-plane 이니 $\sin\theta = 1$ 로 고정. $\psi = dk\cos\phi$

$$A_{net} = A_0(\vec{r})\sum_{n=0}^{4} e^{-iknd\cos\phi}$$

summation 항을 structure factor S 라고 부르고, 계산해 보면,

$$S = \sum_{n=0}^{4} e^{-iknd\cos\phi} = \sum_{n=0}^{4}\left(e^{-ikd\cos\phi}\right)^n = \frac{1-e^{-5\cdot ikd\cos\phi}}{1-e^{-ikd\cos\phi}}$$

등비급수 공식 적용

간단하게 $\psi = kd\cos\phi$ 라고 두고 $|S|^2$을 풀어보자

$$S = \frac{1-e^{-i5\psi}}{1-e^{-i\psi}}$$

$$|1-e^{i\psi}|^2 = (1-e^{-i\psi})(1-e^{i\psi}) = 2-2\cos\psi$$

이때, $1-\cos\psi = \left(\sin^2\frac{\psi}{2}+\cos^2\frac{\psi}{2}\right) - \left(\cos^2\frac{\psi}{2}-\sin^2\frac{\psi}{2}\right) = 2\sin^2\frac{\psi}{2}$ 임을 이용.

$$|1-e^{i\psi}|^2 = 4\sin^2\frac{\psi}{2}$$

$$|S|^2 = \frac{|1-e^{i5\psi}|^2}{|1-e^{i\psi}|^2} = \frac{\sin^2\frac{5}{2}\psi}{\sin^2\frac{1}{2}\psi}$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle \propto \mathrm{Re}\left[r^2 \hat{n} \cdot (\mathbb{E} \times \mathbb{H}^*) \right]$$

$\vec{\mathbb{H}} = \frac{1}{\mu_0} \nabla \times \vec{\mathbb{A}} \simeq ik\hat{r} \times \vec{\mathbb{A}}$

인 것과, far field 에서 $\mathbb{E} \propto i\vec{k} \times \vec{\mathbb{A}}$, $\mathbb{H} \propto i\vec{k} \times \mathbb{E} \propto -\vec{k} \times (\vec{k} \times \vec{\mathbb{A}})$

$\vec{\mathbb{E}} = \frac{i}{k} Z_0 \nabla \times \vec{\mathbb{H}} \simeq -Z_0 \hat{r} \times (ik\hat{r} \times \vec{\mathbb{A}})$

임에 주목. 그리고 문제에서 dipole 하나의 angle dependent radiation power 는 $\frac{dP}{d\Omega} \propto k^4 |\vec{p}|^2 \sin^2\theta$ 라고 한 단서를 이용하자.

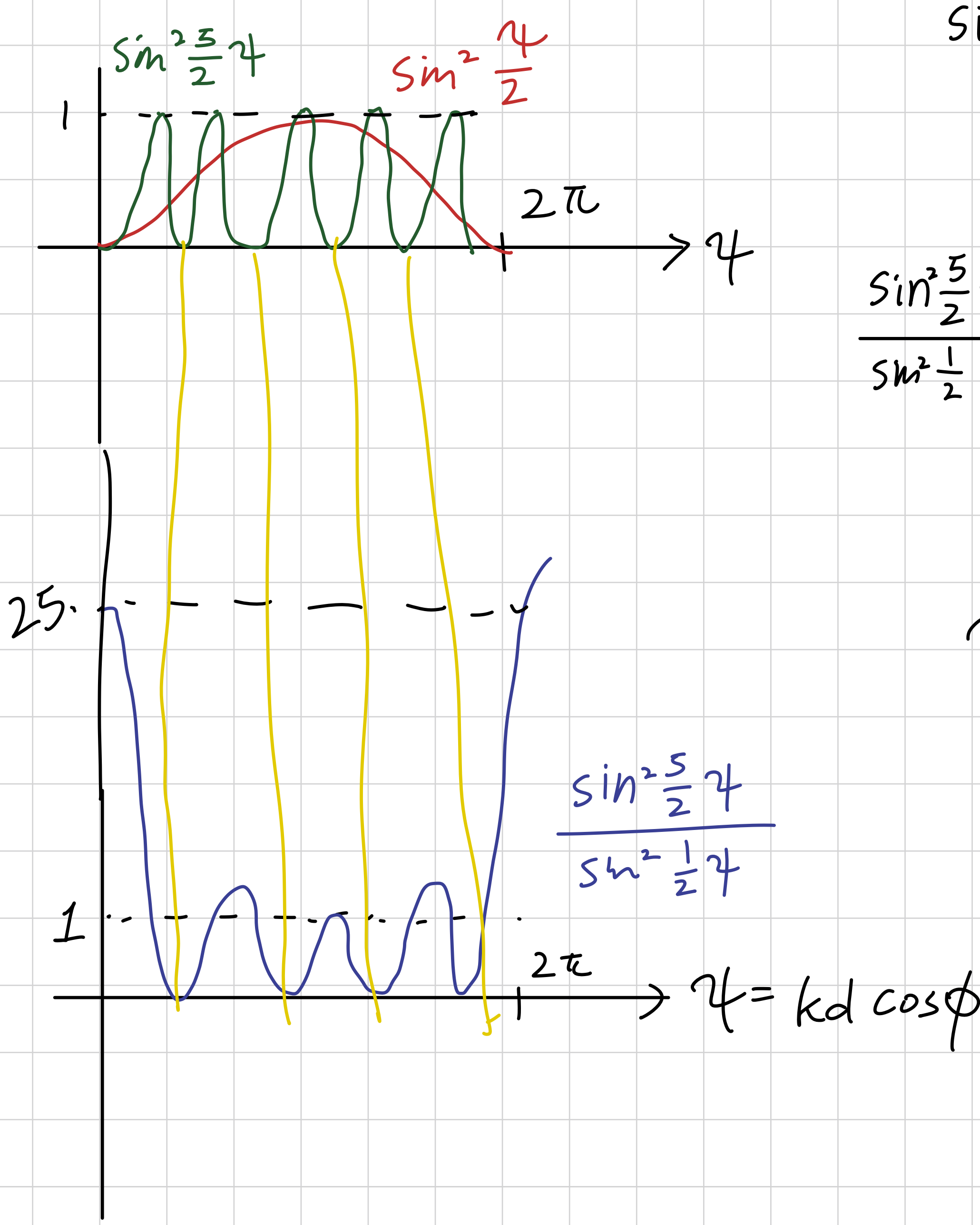
dipole 하나만 있을 때 field 를 $\mathbb{E}_0$, $\mathbb{H}_0$ 라고 볼 때, $N=5$ 여러 dipole 이 있을 때 field $\mathbb{E}_5$, $\mathbb{H}_5$ 는,

$\mathbb{E}_5 = S\mathbb{E}_0$, $\mathbb{H} = S\mathbb{H}_0$ 임이 자명,

$$\left\langle \frac{dP}{d\Omega} \right\rangle \propto \mathrm{Re}\left[r^2 \hat{n} \cdot (\mathbb{E}_5 \times \mathbb{H}_5^*) \right] = |S|^2 \mathrm{Re}\left[r^2 \hat{n} \cdot (\mathbb{E}_0 \times \mathbb{H}_0^*) \right]$$

$$= |S(\theta)|^2 k^4 |\vec{p}|^2 \sin^2 \frac{\pi}{2}$$

$$= \frac{\sin^2 \frac{5}{2}\psi}{\sin^2 \frac{\psi}{2}} k^4 |\vec{p}|^2$$



$\psi = kd\cos\phi$

$$\frac{\sin^2 \frac{5}{2}\psi}{\sin^2 \frac{1}{2}\psi} \simeq \frac{\left[\frac{5}{2}\psi + \frac{1}{2}\left(\frac{5}{2}\right)^2\psi^2 + \cdots\right]^2}{\left[\frac{1}{2}\psi + \frac{1}{2}\left(\frac{1}{2}\right)^2\psi^2 + \cdots\right]^2}$$

$$= \left(\frac{\frac{5}{2}}{\frac{1}{2}}\right)^2 + c\psi^3$$

$\psi \to 0$. $\simeq$ 25

C 풀이)

$\psi = kd\cos\phi$ 가 $2\pi m$ (m is an integer) 일 때 maximum

ψ 가 $2\pi(m + \frac{n}{5})$ ($n = 1$ or 2 or 3 or 4) 일 때 completly suppressed

Lienard - Wichert potential

Lienard-Wiechert potential, fields (3 points)

Start from the scalar and vector potentials in the Lorentz gauge,

$$\varphi(\vec{x},t) = \int dt' \frac{d^3\vec{x}\prime}{|\vec{x}-\vec{x}\prime|}\rho(\vec{x}',t')\delta\left(t' + \frac{|\vec{x}-\vec{x}\prime|}{c} - t\right),$$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int dt' \frac{d^3\vec{x}'}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',t')\delta\left(t' + \frac{|\vec{x}-\vec{x}'|}{c} - t\right),$$

with $\rho(\vec{x},t) = e\delta[\vec{x}-\vec{r}(t')]$, $\vec{J}(\vec{x},t) = ec\vec{\beta}\delta[\vec{x}-\vec{r}(t')]$, and $\frac{\vec{r}(t')}{dt} = c\vec{\beta}$,

1. (0.5) Obtain the Lienard-Wiechert potential, as in (14.18).

$$\varphi(\vec{x},t) = \frac{e}{(1-\hat{n}\cdot\vec{\beta})|\vec{x}-\vec{r}(t')|}\Bigg]_{retarded}$$

$$\vec{A}(\vec{x},t) = \frac{e\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})|\vec{x}-\vec{r}(t')|}\Bigg]_{retarded}$$

2. (0.5) From the definition of the retarded time, $t' = t - \frac{|\vec{x}-\vec{x}\prime|}{c}$, show that $dt' = -\frac{1}{c}\frac{\hat{n}\cdot d\vec{x}}{(1-\hat{n}\cdot\vec{\beta})}$, or $\nabla t' = -\frac{1}{c}\frac{\hat{n}}{(1-\hat{n}\cdot\vec{\beta})}$, for constant time (t) but with variation in the position.
3. (2) By letting $D = (1-\hat{n}\cdot\vec{\beta})|\vec{x}-\vec{r}(t')|$, find the relations for $\frac{\partial D}{\partial t}$ and ∇D. Now, with these relations, find the E and B.

Note that

$$\vec{E}(\vec{x},t) = \frac{e}{D^2}\left(\nabla D + \frac{\vec{\beta}}{c}\frac{\partial D}{\partial t}\right) - \frac{ec\dot{\boldsymbol{\beta}}}{D}\frac{\partial t'}{\partial t}$$

, and

$$\vec{B}(\vec{x},t) = \frac{e}{D^2}\vec{\beta}\times\nabla D + \frac{e}{D}\nabla t'\times\dot{\boldsymbol{\beta}}$$

$$\phi(\vec{x},t) = \int dt' \int d^3\vec{x}' \frac{\rho(\vec{x}',t')}{|\vec{x}-\vec{x}'|} \delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int dt' \int d^3\vec{x}' \frac{\vec{J}(\vec{x}',t')}{|\vec{x}-\vec{x}'|} \delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)$$

$$\rho(\vec{x},t) = e\,\delta[\vec{x}-\vec{r}(t')]$$

$$\vec{J}(\vec{x},t) = ec\vec{\beta}\,\delta[\vec{x}-\vec{r}(t')]$$

$$\frac{\vec{r}(t')}{dt} = c\vec{\beta}$$

1) 풀이 →

$$\phi(\vec{x},t) = \int dt' \int d^3\vec{x}' \frac{e}{|\vec{x}-\vec{x}'|} \delta[\vec{x}-\vec{r}(t')]\,\delta\left(t'-t+\frac{|x-x'|}{c}\right)$$

$$= \int dt' \frac{e}{|\vec{x}-\vec{r}(t')|} \delta\left(t'-t+\frac{|x-\vec{r}(t')|}{c}\right)$$

$t_0'-t+\frac{|x-\vec{r}(t_0')|}{c} = 0$ 을 만족하는 t_0'

그리고 $f(t') = t'-t+\frac{|x-\vec{r}(t')|}{c}$ 일 때.

위 적분결과는 $\frac{1}{|\vec{x}-\vec{r}(t_0')|}\frac{1}{\left|\frac{df}{dt'}\Big|_{t'=t_0'}\right|}$ 이다.

$$\frac{df}{dt'} = 1+\frac{1}{c}\frac{d}{dt'}|\vec{x}-\vec{r}(t')|$$

$$= 1-\hat{n}\cdot\vec{\beta}$$

$$\frac{d}{dt'}|\vec{x}-\vec{r}(t')|$$

$$= \frac{d}{dt'}\left[(\vec{x}-\vec{r}(t'))\cdot(\vec{x}-\vec{r}(t'))\right]^{\frac{1}{2}}$$

$$= \frac{1}{|\vec{x}-\vec{r}(t')|}[\vec{x}-\vec{r}(t')]\cdot\frac{d}{dt'}[-\vec{r}(t')]$$

$$= -\hat{n}\cdot\frac{d}{dt'}\vec{r}(t')$$

$$= -c\hat{n}\cdot\vec{\beta}$$

$$\therefore \phi(\vec{x},t) = \frac{e}{(1-\hat{n}\cdot\vec{\beta})|\vec{x}-\vec{r}(t_0')|}$$

$$\vec{A} = e\vec{\beta} \underline{\int dt' \int d^3\vec{x}' \frac{1}{|\vec{x}-\vec{x}'|} \delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

이미 풀었던 적분이다.

$$= \vec{\beta}\, \phi(\vec{x},t)$$

$$= \frac{e\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})\,|\vec{x}-\vec{r}(t_0')|}$$

2) 풀이 →

$$t' = t - \frac{1}{c}\left|\vec{x}-\vec{x}'(t')\right|$$

$$\frac{dt'}{dt} = 1 - \frac{1}{c}\frac{d}{dt}\left[(\vec{x}-\vec{x}'(t'))\cdot(\vec{x}-\vec{x}'(t'))\right]$$

$$= 1 - \frac{1}{c}\frac{1}{|\vec{x}-\vec{x}'(t')|}(\vec{x}-\vec{x}'(t'))\cdot\frac{d}{dt}(\vec{x}-\vec{x}'(t'))$$

$$= 1 - \frac{1}{c}\hat{n}\cdot\left(\frac{d\vec{x}}{dt} - \frac{dt'}{dt}\frac{d}{dt'}\vec{x}'(t')\right)$$

$$= 1 - \frac{1}{c}\hat{n}\cdot\left(\frac{d\vec{x}}{dt} - \frac{dt'}{dt}c\vec{\beta}\right)$$

$$= 1 + \hat{n}\cdot\vec{\beta}\frac{dt'}{dt} - \frac{1}{c}\hat{n}\cdot\frac{d\vec{x}}{dt}$$

$$(1-\hat{n}\cdot\vec{\beta})dt' = dt - \frac{1}{c}\hat{n}\cdot d\vec{x}$$

$$dt' = \frac{dt - \frac{1}{c}\hat{n}\cdot d\vec{x}}{1-\hat{n}\cdot\vec{\beta}} \xrightarrow{dt=0} = -\frac{1}{c}\frac{\hat{n}\cdot d\vec{x}}{1-\hat{n}\cdot\vec{\beta}}$$

$$\boxed{\begin{aligned} dt' &= (\nabla t')\cdot d\vec{x} \\ \nabla t' &= -\frac{1}{c}\frac{\hat{n}}{1-\hat{n}\cdot\vec{\beta}} \end{aligned}}$$

3-1) 풀이 → $D=(1-\hat{n}\cdot\vec{\beta})|\vec{x}-\vec{r}(t')|$

$D=|\vec{x}-\vec{r}(t')|-(\vec{x}-\vec{r}(t'))\cdot\vec{\beta}$,

잊지 말 것!

$$\boxed{\frac{\partial D}{\partial t}=\frac{\partial t'}{\partial t}\frac{\partial D}{\partial t'}}$$

$$\frac{dt'}{dt}=\frac{1}{1-\hat{n}\cdot\vec{\beta}}$$

$$\frac{d}{dt'}|\vec{x}-\vec{r}(t')|=\frac{(\vec{x}-\vec{r}(t'))}{|\vec{x}-\vec{r}(t')|}\cdot\underbrace{\frac{d}{dt}(\vec{x}-\vec{r}(t'))}_{-c\vec{\beta}}=-c\,\hat{n}\cdot\vec{\beta}$$

$$\frac{\partial t'}{\partial t}=1+\hat{n}\cdot\vec{\beta}\frac{dt'}{dt}\cancel{-\frac{1}{c}\hat{n}\cdot\frac{d\vec{x}}{dt}}$$

$$\frac{d}{dt'}(\vec{x}-\vec{r}(t'))\cdot\vec{\beta}=\underbrace{\left[\frac{d}{dt}(\vec{x}-\vec{r}(t'))\right]}_{-c\vec{\beta}}\cdot\vec{\beta}+(\vec{x}-\vec{r}(t'))\cdot\dot{\vec{\beta}}=-c|\beta|^2+(\vec{x}-\vec{r}(t'))\cdot\dot{\vec{\beta}}$$

$$=-c|\beta^2|+|\vec{x}-\vec{r}(t')|\hat{n}\cdot\vec{\beta}$$

$-c\,\hat{n}\cdot\vec{\beta}$

$$\frac{\partial D}{\partial t'}=-c\hat{n}\cdot\vec{\beta}+c\ \beta^2-|\vec{x}-\vec{r}(t')|\,\hat{n}\cdot\vec{\beta}$$

$$\frac{\partial D}{\partial t}=\frac{1}{1-\hat{n}\cdot\vec{\beta}}\left\{c(\beta^2-\hat{n}\cdot\vec{\beta})-|\vec{x}-\vec{r}(t')|\,\hat{n}\cdot\vec{\beta}\right\}$$

계산 결과 : $\frac{\partial D}{\partial t}=\frac{c(\beta^2-\hat{n}\cdot\vec{\beta})-|\vec{x}-\vec{r}(t')|\hat{n}\cdot\vec{\beta}}{1-\hat{n}\cdot\beta}$

$$\left.\frac{\partial D}{\partial \vec{x}}\right|_{t'}=\frac{\partial}{\partial\vec{x}}|\vec{x}-\vec{r}(t')|-\left(\frac{\partial}{\partial\vec{x}}(\vec{x}-\vec{r}(t'))\right)\cdot\vec{\beta}$$

$$=\hat{n}-\vec{\beta}$$

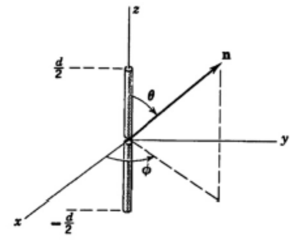
$$\nabla D=\left(-\frac{1}{c}\frac{\hat{n}}{1-\hat{n}\cdot\vec{\beta}}\right)\left\{c(\beta^2-\hat{n}\cdot\vec{\beta})-|\vec{x}-\vec{r}(t')|\,\hat{n}\cdot\vec{\beta}\right\}$$

$$+\hat{n}-\vec{\beta}$$

$$\boxed{\nabla D=\frac{\partial D}{\partial t'}\nabla t'+\left.\frac{\partial D}{\partial\vec{x}}\right|_{t'}}$$

$$\nabla D=\frac{(1-\beta^2)\hat{n}}{1-\hat{n}\cdot\vec{\beta}}+\frac{|\vec{x}-\vec{r}(t')|(\hat{n}\cdot\dot{\vec{\beta}})\hat{n}}{c(1-\hat{n}\cdot\vec{\beta})}-\vec{\beta}$$

4. (50 points) Radiation from a center-fed linear antenna. Find the angular distribution of radiation power in terms of antenna length (d), current (I), and the wavelength ($1/k$).



Hint: you can use the angular distribution from an electric dipole moment:

$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |\mathbf{p}|^2 \sin^2\theta$

Note that the current in the dipole radiator is expressed as

$$I(z)e^{-i\omega t} = I_0\left(1 - \frac{2|z|}{d}\right)e^{-i\omega t}$$

$I(z,t) = e^{-i\omega t} I(z)$

$I(z) = I_0\left(1 - \frac{2|z|}{d}\right)$, $J(\vec{x}) = I_0\,\delta(x)\,\delta(y)\left(1 - \frac{2|z|}{d}\right)$

연속방정식, $\frac{d\rho}{dt} + \nabla\cdot J = 0$

이때, $\rho(x,t) = e^{-i\omega t}\rho(x)$ 라면, $\dot{\rho} = -i\omega\,\rho(x,t)$

$-i\omega\,\rho(\vec{x}) + \nabla\cdot\left\{\underbrace{I_0\,\delta(x)\,\delta(y)\left(1 - \frac{2|z|}{d}\right)}_{J(\vec{x})}\right\} = 0$

$\rho = \frac{i}{\omega}\nabla\cdot J(x)$

dipole moment, $i\omega\vec{p}(x) = \int d^3\vec{x}\; x\,\rho(\vec{x})$

$\vec{P}(x) = \frac{i}{\omega}\int x\,\nabla\cdot J(x)$

$= \int d^3\vec{x}\; x\,\nabla\cdot J(\vec{x})$

$= \left[x J\right]_{(0,0,-\frac{1}{2})}^{(0,0,\frac{1}{2})} - \int d^3\vec{x}\, J$

$= -\frac{i}{\omega}$

$= -\int d^3\vec{x}\, J(\vec{x})$

$\vec{p} = \frac{i}{\omega}\int d^3\vec{x}\, J(x) = \frac{iI_0}{\omega}\int d^3\vec{x}\,\delta(x)\,\delta(y)\left(1 - \frac{2|z|}{d}\right) = \frac{iI_0}{\omega}\int_{-\frac{d}{2}}^{\frac{d}{2}}\left(1 - \frac{2|z|}{d}\right)dz\,\hat{z}$

$= \frac{2iI_0}{\omega}\int_0^{\frac{d}{2}}\left(1 - \frac{2z}{d}\right)dz\,\hat{z}$

$= \frac{2iI_0}{\omega}\left[z - \frac{1}{d}z^2\right]_0^{\frac{d}{2}}\hat{z}$

$= \frac{iI_0}{\omega}\cdot 2\cdot\frac{d}{2}\left(1 - \frac{1}{d}\cdot\frac{d}{2}\right)$

$= \frac{iId}{2\omega}\hat{z}$

Dipole moment of angular radiation power : $\frac{dP}{d\Omega} = \frac{c^2 Z_0}{32\pi^2} k^4 |p|^2 \sin\theta$

$$C = \frac{\omega}{k}, \qquad \omega = Ck.$$

$$\vec{p} = \frac{i I_0 d}{2\omega}\hat{z} = \frac{i I_0 d}{2Ck}\hat{z}. \qquad |\vec{p}|^2 = \frac{I_0^2 d^2}{4C^2 k^2}$$