# Spherical harmonics

$$\langle x | \alpha, l, m \rangle = R_a(r)\, Y_l^m(\theta, \phi)$$

$$Y_l^m \propto P_l^m(\cos\theta)\, e^{im\phi}$$

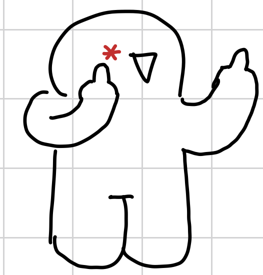
$m>0$ 이면
$\phi$ 가 커짐에 따라
$+z$ 술로 돌아감.
직관적인 부분

$$P_l^m \propto \sin^{-m}\theta \left(\frac{d}{d\cos\theta}\right)^{l-m} (\sin\theta)^{2l}$$

## Probability current density

$$J = \frac{\hbar}{m} \mathrm{Im}[\psi^* \nabla \psi]$$

각속도 차원, $[T^{-1}]$



$$Y_l^{\pm l} \propto e^{\pm il\phi} (\sin\theta)^l$$

$$\langle \hat{n} | L_z | \alpha \rangle = -i\hbar \frac{\partial}{\partial \phi} \langle \hat{n} | \alpha \rangle$$

$\theta$와 $\phi$의 정보

$$\langle \hat{x} | p_x | \alpha \rangle = -i\hbar \frac{\partial}{\partial x} \langle \hat{x} | \alpha \rangle$$

$$\langle \hat{n} | L_x | \alpha \rangle = i\hbar \left( \sin\phi \frac{\partial}{\partial \theta} + \cos\phi \cot\theta \frac{\partial}{\partial \phi} \right) \langle \hat{n} | \alpha \rangle$$

$$\langle \hat{n} | L_y | \alpha \rangle = i\hbar \left( -\cos\phi \frac{\partial}{\partial \theta} + \sin\phi \cot \frac{\partial}{\partial \phi} \right) \langle \hat{n} | \alpha \rangle$$

$\sin^2\theta$

$$\langle \hat{n} | L^2 | \alpha \rangle = -\hbar^2 \left( \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) \right) \langle \hat{n} | \alpha \rangle$$

$$Y_l^m(\phi, \theta) \propto D_{m0}^{l}(\alpha=\phi, \beta=\theta)$$

$$Y_l^m(\phi, \theta) = \sqrt{\frac{2l+1}{4\pi}}\, D_{m0}^{l}(\alpha=\phi, \beta=\theta)$$

$$Y_l^m(\theta, \phi) = \langle \hat{n} | l m \rangle$$

$Y_l^m(\theta, \phi)$ 에 대한 미분 방정식 : $L_z$와 $L^2$의 원래 기능과 미분형식을 이용.

① $L_z$ 이용. $\langle \hat{n} | L_z | l m \rangle = m\hbar\, Y_l^m(\theta, \phi) = -i\hbar \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi)$

$$\frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) = im\, Y_l^m(\theta, \phi)$$

$Y_l^m$ 에서 $\theta$ 관련 부분은 $\propto e^{im\phi}$ 임.

$$Y_l^m = e^{im\phi}\, y_l^m(\theta)$$

② $L^2$ 이용. $\langle \hat{n} | L^2 | l m \rangle = \hbar^2 l(l+1)\, Y_l^m(\theta, \phi)$

$$= -\hbar^2 \left( \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) \right) Y_l^m(\theta, \phi)$$

$$\left( \frac{1}{\sin^2\theta} \frac{\partial}{\partial \phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\right) + l(l+1) \right) Y_l^m(\theta, \phi) = 0$$

## 푸아송 괄호와 Commutator

$$\boxed{[G,\mathcal{H}] = i\hbar\{G,\mathcal{H}\}}$$

ħ가 곱해져야 물리적 차원이 맞음

$[x,p] = i\hbar$ 는 이미 알고 있지?

$\{x,p\} = \frac{\partial x}{\partial x}\frac{\partial p}{\partial p} - \frac{\partial x}{\partial p}\frac{\partial p}{\partial x} = 1$.

## Heisenberg equation of motion

$$\frac{dG}{dt} = \frac{1}{i\hbar}[G,\mathcal{H}] + \frac{\partial G}{\partial t}$$

## EOM with poisson bracket

$$\frac{dG}{dt} = \left(\frac{\partial G}{\partial q}\right)\dot{q} + \left(\frac{\partial G}{\partial p}\right)\dot{p} + \frac{\partial G}{\partial t}$$

$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}$

$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q}$

$$= \frac{\partial G}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial G}{\partial p}\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial G}{\partial t}$$

$$\frac{\partial G}{dt} = \{G,\mathcal{H}\} + \frac{\partial G}{\partial t}$$

## Energy eigenket의 time-evolution

$\mathcal{H}|n\rangle = E_n|n\rangle$

$|n,t\rangle = \exp(-\frac{i}{\hbar}\mathcal{H}t)|n\rangle = \exp(-i\frac{E_n}{\hbar}t)|n\rangle$

$\therefore \omega_n = \frac{E_n}{\hbar}$

임의의 상태, $|\alpha\rangle = \sum_n C_n|n\rangle$

$|\alpha,t\rangle = \sum_n \exp(-i\omega_n t)|n\rangle$

## Hermitian : † 와 자신이 같다. 관측값

$$A = A^\dagger$$

## Unitary : † 와 inverse가 같다.

Transformation

$$AA^\dagger = \mathbb{1}$$

왜 Transformation은 Unitary?

⟶ 내적을 보존하기 위해!

$\langle\alpha|\alpha\rangle = \langle\alpha'|\alpha'\rangle = \langle\alpha|\underbrace{A^\dagger A}_{\mathbb{1}}|\alpha\rangle$.

## SU(2)

$$U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad |a|^2+|b|^2=1$$

⟶ $U^\dagger U = 1$ 나옴

$a$ 와 $b$ 는 complex number, $a = \alpha+i\beta$, $b = \gamma+i\delta$

$\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$. → 4 차원 공간에서 구껍질.

이제 양자공부 시작! 야호.

infinitesimal angle

$D_{\hat{n}}(\epsilon) = 1 - \frac{i}{\hbar}\epsilon\, \hat{n}\cdot J$

이래야 차원이 무차원 되거든

continuous angle $\phi$ ⟶ finite rotation

$D_{\hat{n}}(\phi) = \exp\left(-\frac{i}{\hbar}\phi\, \hat{n}\cdot J\right) = \lim_{N\to\infty}\left(1-\frac{i}{\hbar}\frac{\phi}{N}\hat{n}\cdot J\right)^N$

Relation between generators (G) and operators (O)

각운동량 차원

$$O(\lambda) = \lim_{N\to\infty}\left[1 - \frac{i}{\hbar}\frac{\lambda}{N}G\right]^N = \exp\left(-\frac{i}{\hbar}\lambda G\right)$$

⟨Baker-Hausdorff lemma⟩

notation : $\underbrace{[G,[G,[G\cdots,[G}_{n\text{번}},A]\cdots]]]$

$= [^nG, A]$

and, $[^0G,A]=A$

$$O^\dagger(\lambda) A O(\lambda) = \exp\left(\frac{i}{\hbar}\lambda G\right)A\exp\left(-\frac{i}{\hbar}\lambda G\right)$$

$$= \sum_n^\infty \frac{1}{n!}\left(\frac{i}{\hbar}\lambda\right)^2 [^nG,A]$$

$$= A + \frac{i}{\hbar}\lambda[G,A] + \frac{1}{2}\left(\frac{i}{\hbar}\lambda\right)^2[G,[G,A]]$$

$$e^{i\theta} = \underbrace{1}_{\cos\theta} + \underbrace{i\theta}_{i\sin\theta} + \underbrace{\frac{(i)^2}{2}\theta^2}_{\cos\theta} + \underbrace{\frac{(i)^3}{3!}\theta^3}_{i\sin\theta} + \underbrace{\frac{(i)^4}{4!}\theta^4}_{\cos\theta}\cdots$$

$\cos\theta = \sum_{n\text{ is even}} \frac{1}{n!}(i\theta)^n$ $\quad\sin\theta = -i\sum_{m\text{ is odd}}\frac{1}{n!}(i\theta)^n$

$[J_i, J_j] = i\hbar\,\varepsilon_{ijk}J_k$ $\quad [J^2, J_i] = 0$

$J_\pm = J_x \pm iJ_y$ → $J_+$ 와 $J_-$ 는 † 관계

$[J_+, J_-] = -2i[J_x, J_y] = 2\hbar J_z$

$[J_z, J_\pm] = \pm\hbar J_\pm$

$J_z(J_\pm|l\, m\rangle) = \{[J_z,J_\pm] + J_\pm J_z\}|l\, m\rangle$

$= \{\pm\hbar J_\pm + m\hbar J_\pm\}|l\, m\rangle$

$= \hbar\{m\pm 1\}J_\pm|l\, m\rangle$

$J^2|a\, b\rangle = a|ab\rangle$ $\quad J_z|a\, b\rangle = b|a\, b\rangle$

a와 b에 대해 알아 내자.

① $a - b^2 \geq 0$ 증명

$J^2 - J_z^2 = \frac{1}{2}\{J_+, J_-\} = \frac{1}{2}(J_-^\dagger J_- + J_+^\dagger J_+)$

를 $|a\, b\rangle$ 에 샌드위치,

② $b_{max}$ 와 $b_{min}$ 결정, a와 관계식

$J_+|a\, b_{max}\rangle = 0,\quad J_-J_+|a\, b_{max}\rangle = 0$

$J_-J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i\underbrace{[J_x, J_y]}_{i\hbar J_z}$

$= J^2 - J_z^2 - \hbar J_z$

$a^2 - b_{max}^2 - \hbar b_{max} = 0$ $\quad a = b_{max}(b_{max}+\hbar)$

$C^+_{jm} = \hbar\sqrt{(j-m)(j+m+1)}$ → 유도: $J_-J_+ = J_+^\dagger J_+$ 를 두 가지 방식으로 적용.

$C^-_{jm} = \hbar\sqrt{(j+m)(j-m+1)}$

$\frac{1}{2}$ spin state $|\alpha\rangle$ 을 z축에 대해 $\phi$ 만큼 돌리기

$\exp\left(-\frac{i}{\hbar}\phi J_z\right)|\alpha\rangle$

$= \exp\left(-\frac{i}{\hbar}\phi J_z\right)\{|+\rangle\langle+|\alpha\rangle + |-\rangle\langle-|\alpha\rangle\}$

$= \exp\left(-i\frac{\phi}{2}\right)|+\rangle\langle+|\alpha\rangle + \exp\left(+i\frac{\phi}{2}\right)|-\rangle\langle-|\alpha\rangle$

$\frac{1}{2}$이 붙은 게 특징적. $\quad\phi = 2\pi$ 를 넣으면 위상이 반대

$= -|+\rangle\langle+|\alpha\rangle - |-\rangle\langle-|\alpha\rangle = -|\alpha\rangle$

# $\vec{J}$ 의 vector property

$\longrightarrow \langle \vec{J} \rangle$ 가 vector 처럼 회전에 변환

문제) z축에 대해 $\phi$ 회전,

$$\begin{pmatrix} \langle J_x' \rangle \\ \langle J_y' \rangle \\ \langle J_z' \rangle \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \langle J_x \rangle \\ \langle J_y \rangle \\ \langle J_z \rangle \end{pmatrix}$$

임을 보여라.

① $D_z^\dagger(\phi) J_x D_z(\phi) = \cos\phi J_x - \sin\phi J_y$

인 걸 보이자.

$$\exp\left(i\frac{\phi}{\hbar}J_z\right) J_x \exp\left(-i\frac{\phi}{\hbar}J_z\right) = \sum_n \frac{1}{n!}\left(\frac{i\phi}{\hbar}\right)^n [{}^n J_z, J_x]$$

$[{}^0 J_z, J_x] = J_x \longrightarrow 1$

$[{}^1 J_z, J_x] = i\hbar J_y \rightarrow i$

$[{}^2 J_z, J_x] = -(i\hbar)^2 J_x \rightarrow 1$

$[{}^3 J_z, J_x] = -(i\hbar)^3 J_y \rightarrow i$

$\hbar^n$ 을 제외한 계수의 패턴

$$[{}^n J_z, J_x] = \begin{cases} \hbar^n J_x & \text{if } n \text{ is even} \\ i\hbar^n J_y & \text{if } n \text{ is odd} \end{cases}$$

$$\sum_n \left(\frac{i\phi}{\hbar}\right)^n [{}^n J_z, J_x] = \underbrace{\sum_{m\ \text{even}} \frac{1}{m!}(i\phi)^m J_x}_{\substack{J_x\ \exp(i\phi)\text{의 even part} \\ \to J_x \cos\phi}} + \underbrace{\sum_{k\ \text{odd}} \frac{i}{k!}(i\phi)^k J_y}_{\substack{iJ_y\ \exp(i\phi)\text{의 odd part.} \\ \to -J_y \sin\phi}}$$

$\therefore \langle J_x' \rangle = \langle J_x \rangle \cos\phi - \langle J_y \rangle \sin\phi$

② $D_z^\dagger(\phi) J_y D_z(\phi) = \sin\phi J_x + \cos\phi J_y$

임을 보여라.

$\sum_n \frac{1}{n!}\left(i\frac{\phi}{\hbar}\right)^n [{}^n J_z, J_y]$ 를 분석

$[{}^0 J_z, J_y] = J_y \rightarrow 1$

$[{}^1 J_z, J_y] = -i\hbar J_x \rightarrow -i$

$[{}^2 J_z, J_y] = -(i\hbar)^2 J_y \rightarrow 1$

$[{}^3 J_z, J_y] = (i\hbar)^3 J_x \rightarrow -i$

$\hbar^n$ 을 제외한 계수 패턴

$$[{}^n J_z, J_y] = \begin{cases} \hbar^n J_y & n \text{ is even} \\ -i\hbar^n J_x & n \text{ is odd} \end{cases}$$

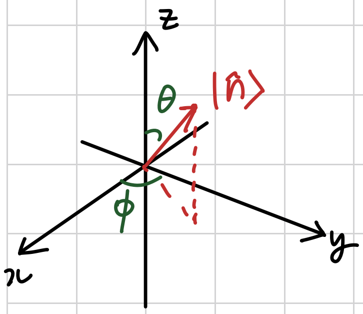
$$\sum_n \frac{1}{n!}\left(i\frac{\phi}{\hbar}\right)^n [{}^n J_z, J_y] = \underbrace{\sum_{m\ \text{even}} \frac{1}{n!}(i\phi)^n J_y}_{\substack{J_y\ \exp(i\phi)\text{의 even part,} \\ J_y \cos\phi}} + \underbrace{\sum_{k\ \text{odd}} -i\frac{1}{n!}(i\phi)^n J_x}_{\substack{-iJ_x\ \exp(-i\phi)\text{의 odd part,} \\ J_x \sin\phi}}$$

$\langle J_y \rangle' = \langle J_y \rangle \cos\phi + \langle J_x \rangle \sin\phi$

이렇게 even /odd 로 나누어 삼각함수를 뽑는 풀이가 많이나옴

# Time reversal for $\frac{1}{2}$-spin system → 시험 나올 느낌?

$\hat{n}$ 방향의 spinor $|\hat{n},\pm\rangle$ 를
$|\hat{z},\pm\rangle$ 로부터 유도




$|\hat{z},\pm\rangle$ 를
y축에 대해 $\theta$ 돌리고,
z 축에 대해 $\phi$ 돌린다.

$$|\hat{n},\pm\rangle = \exp\left(-\frac{i}{\hbar}\phi S_z\right)\exp\left(-\frac{i}{\hbar}\theta S_y\right)|z,\pm\rangle$$

$|\hat{n},\pm\rangle$ 에 $\Theta$을 적용. $S$도 TR-odd

$$\Theta|\hat{n},\pm\rangle = \exp\left(\frac{i}{\hbar}\phi(-S_z)\right)\exp\left(\frac{i}{\hbar}\theta(-S_y)\right)\Theta|z,\pm\rangle$$

$$= \exp\left(-\frac{i}{\hbar}\phi S_z\right)\exp\left(-\frac{i}{\hbar}\theta S_y\right)\Theta|z,\pm\rangle$$

$\Theta|z,\pm\rangle$ 는 뭔가?

$\Theta S_z = -S_z\Theta$ 인 걸 이용.

$$S_z\Theta|\hat{z},\pm\rangle = -\Theta S_z|\hat{z},\pm\rangle = \mp\frac{\hbar}{2}\Theta|\hat{z},\pm\rangle$$

즉, $\Theta|\hat{z},\pm\rangle$ 의 $S_z$ 에 대한 eigen value 는 $\mp\frac{\hbar}{2}$
스핀이 $\Theta$ 에 의해 뒤집힌 꼴
Phase factor $\eta$ 을 감안,

$$\Theta|\hat{z},\pm\rangle = \eta|\hat{z},\mp\rangle$$

$$\Theta|\hat{n},\pm\rangle = \eta|\hat{n},\mp\rangle$$

$|\hat{n},\mp\rangle$ 를 만드는 다른 방법, $|\hat{z},\pm\rangle$ 를 돌려 만드는데
y축에 대해 $\theta+\pi$ 돌리고 $\hat{z}$ 에 대해 $\phi$ 돌리기

$$|\hat{n},\mp\rangle = \exp\left(-i\frac{\phi}{\hbar}S_z\right)\exp\left(-i\frac{\theta+\pi}{\hbar}S_y\right)|\hat{z},\pm\rangle$$

$\Theta$ 가 y축에 대해 $\pi$ 더 돌리는 연산이랑 연관,

$$|\hat{n},\mp\rangle = \frac{1}{\eta}\Theta|\hat{n},\pm\rangle = \frac{1}{\eta}\exp\left(-i\frac{\phi}{\hbar}S_z\right)\exp\left(-\frac{\theta}{\hbar}S_y\right)U\underbrace{K|\hat{z},\pm\rangle}_{=|\hat{z},\pm\rangle}$$

$$= \exp\left(-i\frac{\phi}{\hbar}S_z\right)\exp\left(-i\frac{\theta+\pi}{\hbar}S_y\right)|\hat{z},\pm\rangle$$

$$\frac{1}{\eta}U = \exp\left(-i\frac{\pi}{\hbar}S_y\right)$$

$$\Theta = UK = \eta\exp\left(-i\frac{\pi}{\hbar}S_y\right)K$$ in $\frac{1}{2}$-spin system

$$= -\frac{i\eta}{\hbar}2S_yK$$

# $\frac{1}{2}$ - spin system

## Pauli matrix

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

* Hermitian $\sigma^\dagger = \sigma$
* orthogonal ( Unitery )
  $\sigma^\dagger \sigma = \mathbb{1}$
* $(\sigma)^2 = \mathbb{1}$
* $\det(\sigma) = -1$
* $Tr(\sigma) = 0$
* commutation relation
  $[\sigma_i, \sigma_j] = 2i\, \varepsilon_{ijk} \sigma_k$

$$\sigma_+ = \sigma_x + i\sigma_y = 2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma_- = \sigma_x - i\sigma_y = 2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$\frac{1}{2}$-spin operator

$S_i = \frac{\hbar}{2}\sigma_i$ → $\frac{1}{2}$ system의 unit angular momentum 곱함.

spin-rotation operator

$$\exp\left(-\frac{i\varepsilon\, \hat{n}\cdot \vec{J}}{\hbar}\right) = \exp\left(-\frac{i\varepsilon\, \hat{n}\cdot\vec{\sigma}}{2}\right)$$

$[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$ ⟶ J랑 성질 같음

$$(\vec{a}\cdot\vec{\sigma})^2 = |\vec{a}|^2$$

$(\vec{n}\cdot\vec{\sigma})^2 = 1$ . for every 1.

$$\vec{a}\cdot\vec{\sigma} = \begin{pmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{pmatrix}$$

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \mathbb{1}\,\vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$(\hat{n}\cdot\vec{\sigma})^2 = 1$ 을 이용해

$\exp(\frac{i}{\hbar}\phi\,\hat{n}\cdot\mathbf{J}) = \exp(\frac{i}{2}\phi\,\hat{n}\cdot\vec{\sigma})$ 을 직접 구할 수도 있다.

$$\exp\left(-\frac{i}{2}\phi\hat{n}\cdot\vec{\sigma}\right) = \sum_n \frac{1}{n!}\left(-i\frac{\phi}{2}\right)^n (\hat{n}\cdot\vec{\sigma})^n$$

$$= \sum_{m\ \text{even}} \frac{1}{m!}\left(-i\frac{\phi}{2}\right)^m + \sum_{k\ \text{odd}} \frac{1}{n!}\left(-i\frac{\phi}{2}\right)^k \hat{n}\cdot\vec{\sigma}$$

$$= \mathbb{1}\cos\left(\frac{\phi}{2}\right) - i\sin\left(\frac{\phi}{2}\right)(\hat{n}\cdot\vec{\sigma})$$

$\phi = 2\pi$ 를 넣어 봐라. $\mathcal{D}_{\hat{n}}(2\pi) = -\mathbb{1}$ 이 나옴.

$$\hat{n}\cdot\vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z = \begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$\exp\left(-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}\right) = \begin{pmatrix} \cos(\frac{\phi}{2}) - i n_z \sin\frac{\phi}{2} & (-i n_x - n_y)\sin\frac{\phi}{2} \\ (-i n_x + n_y)\sin\frac{\phi}{2} & \cos(\frac{\phi}{2}) + i n_z \sin\frac{\phi}{2} \end{pmatrix}$$

이렇게 임의 방향의 스핀 회전 연산자를
명시적으로 구할 수 있다.

$$\mathcal{D}_z(\phi)\mathcal{D}_y(\theta) = \exp\left(-i\frac{\phi}{2}\sigma_z\right)\exp\left(-i\frac{\theta}{2}\sigma_y\right)$$

$$= \begin{pmatrix} \underbrace{\cos\frac{\phi}{2} - i\sin\frac{\phi}{2}}_{e^{-i\frac{\phi}{2}}} & 0 \\ 0 & \underbrace{\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}}_{e^{i\frac{\phi}{2}}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} & -\sin\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} & \cos\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}$$

임의 방향의
스핀도 |줄 ±>
basis로 나타냄.

$$\therefore |\hat{n}, \pm\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}$$

⇝ Bloch sphere 에
$e^{-i\frac{\phi}{2}}$ 가 곱해진게 나옴.

$S_y$ 로 $\pi$ 만큼 돌려 spin을 뒤집는 연산은?

$$\exp\left(-i\frac{\pi}{2}\sigma_y\right) = \mathbb{1}\cos\left(\frac{\pi}{2}\right) - i\,\sigma_y\sin\left(\frac{\pi}{2}\right)$$

$$= -\frac{i}{\hbar}2S_y$$

# Euler angle

① z축 α ② y'축 β ③ z'축 γ

$$D(\alpha,\beta,\gamma) = D_{z'}(\gamma)\, D_{y'}(\beta)\, D_z(\alpha)$$

$D_{y'}$ 와 $D_{z'}$ 를 없애고 싶다!

① $D_{y'}(\beta) D_z(\alpha) = D_z(\alpha) D_y(\beta)$

$\longrightarrow D_{y'}(\beta) = D_z(\alpha) D_y(\beta) D_z(-\alpha)$

② $D_{z'}(\gamma) D_{y'}(\beta) = D_{y'}(\beta)\; D_z(\gamma)$

$\longrightarrow D_{z'}(\gamma) = D_{y'}(\beta) D_z(\gamma) D_{y'}(-\beta)$

② 대입, ① 대입하면,

$$D(\alpha,\beta,\gamma) = D_z(\alpha) D_y(\beta) D_z(\gamma)$$

α와 γ의 순서가 바뀐 결과가.

α, β, γ 순서 그대로 표현식이 쓰인다.

이제 $\frac{1}{2}$-spin system Rotation matrix를

완전히 구했다.

$$D^{j=\frac{1}{2}}_{m'm}(\alpha,\beta,\gamma) = \left\langle \tfrac{1}{2}\, m' \right| \exp\left(-i\tfrac{\alpha}{\hbar}S_z\right) \exp\left(-i\tfrac{\beta}{\hbar}S_y\right) \exp\left(-i\tfrac{\gamma}{\hbar}S_z\right) \left| \tfrac{1}{2}\, m \right\rangle$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}} \sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}} \cos\frac{\beta}{2} \end{pmatrix}$$

$$D_z(\gamma) = \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}$$

$$D_x(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & -i\sin\frac{\phi}{2} \\ -i\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$

$$D_y(\beta) = \begin{pmatrix} \cos\frac{\phi}{2} & -\sin\frac{\phi}{2} \\ \sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}$$

$$\exp\left(-i\frac{\phi}{2}\hat{n}\cdot\vec{\sigma}\right) = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) - in_z\sin\frac{\phi}{2} & (-in_x - n_y)\sin\frac{\phi}{2} \\ (-in_x + n_y)\sin\frac{\phi}{2} & \cos\left(\frac{\phi}{2}\right) + in_z\sin\frac{\phi}{2} \end{pmatrix}$$

# Rotation matrix

## Definition

$$D^{(j)}_{m'm}(\hat{n},\phi) = \langle j m' | \exp(-i\frac{\phi}{\hbar}\hat{n}\cdot\vec{J}) | j m\rangle$$

Rotation 한 이후의 상태의 coefficient로서 기능

$$D_{\hat{n}}(\phi)|jm\rangle = \sum_{m'=-j}^{j} D^{(j)}_{m'm}(\hat{n},\phi)|jm'\rangle$$

Euler angle 적용 → $d^{j}_{m'm}(\beta)$ matrix

→ y-axis rotation에 관한 정보.

$$D^{(j)}_{m'm}(\alpha,\beta,\gamma)$$

$$= \underbrace{\langle j m' | \exp(-i\frac{\alpha}{\hbar}J_z)}_{\exp(-i\alpha m')} \exp(-i\frac{\beta}{\hbar}J_y) \underbrace{\exp(-i\frac{\gamma}{\hbar}J_z)|jm\rangle}_{\exp(-i\gamma m)|jm\rangle}$$

$$= e^{-i\alpha m' - i\gamma m} \underbrace{\langle jm' | \exp(-i\frac{\beta}{\hbar}J_y)|jm\rangle}_{d^{j}_{m'm}(\beta)}$$

## j=1, 3차원에서 $d^{1}_{m'm}(\beta)$ 구하기

전략 ① $J_y$의 3×3 matrix expression 찾기

② $\exp(-i\frac{\beta}{\hbar}J_y) = \sum_n \frac{1}{n!}(i\beta)^n(\frac{J_y}{\hbar})^n$

전개, $\left(\frac{J_y}{\hbar}\right)^3 = \frac{J_y}{\hbar}$ 의 주기성 이용.

cosβ, sinβ 성분 나옴.

① 을 위해, $J_y = \frac{1}{2i}(J_+ - J_-)$

$\hbar\sqrt{(j-m)(j+m+1)} = \hbar\sqrt{2}$

$$J_+ = \begin{pmatrix} 0 & C^+_{j=1,m=0} & 0 \\ 0 & 0 & C^+_{j=1,m=-1} \\ 0 & 0 & 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\hbar\sqrt{2}$

$$J_- = \begin{pmatrix} 0 & 0 & 0 \\ C^-_{l=1,m=1} & 0 & 0 \\ 0 & C^-_{l=1,m=0} & 0 \end{pmatrix} = \hbar\sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$\hbar\sqrt{(l+m)(l-m+1)} = \hbar\sqrt{2}$

$$J_y = \frac{1}{2i}\hbar\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

편의상 $\Omega = \frac{J_y}{\hbar}$

$$\Omega^2 = -\frac{1}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}, \quad \Omega^3 = \Omega$$

$$\exp(-\frac{i}{\hbar}\beta J_y) = \exp(-i\beta\Omega)$$

$$= \sum_n \frac{1}{n!}(i\beta)^n\Omega^n = \mathbb{1} + \sum_{\substack{m=2 \\ \text{even}}} \frac{(i\beta)^m}{m!}\Omega^2 + \sum_{k\,\text{odd}} \frac{(i\beta)^k}{k!}\Omega$$

→ 조심!

$$= \mathbb{1} + \Omega^2(\cos\beta - \mathbb{1}) + i\Omega\sin\beta$$

정리, 는 곧이...

# GC coefficients

2 개의 회전, 두 가지 표현법

$|j_1\, j_2\, ;\, j\, m\rangle$ and $|j_1\, j_2\, ;\, m_1\, m_2\rangle$

이 거를

이거의 합으로 어떻게? 나타내는가?
→ CG coefficients

$$|j_1 j_2 ; j\, m\rangle = \sum_{m_1}\sum_{m_2} |j_1 j_2 ; m_1 m_2\rangle \underbrace{\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j\, m\rangle}_{\text{CG coefficient.}}$$

operators : total angular momentum : $\vec{J} = \vec{J}_1 + \vec{J}_2$

z- angular momentum: $J_z = J_{z_1} + J_{z_2}$

$J_+$로 recursion relation 찾기.

$$J_+ |j_1 j_2 ; j\, m\rangle = \sqrt{(j-m)(j+m+1)}\, |j_1 j_2 ; j\, m+1\rangle$$

$$= \sum_{m_1}\sum_{m_2} \underbrace{(J_{+1}+J_{+2})|j_1 j_2 ; m_1 m_2\rangle}_{\left(\sqrt{(j_1-m_1)(j_1+m_1+1)}\,|j_1 j_2 ; m_1+1\ m_2\rangle + \sqrt{(j_2-m_2)(j_2+m_2+1)}\,|j_1 j_2 ; m_1,\ m_2+1\rangle\right)} \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j\, m\rangle$$

$\to \delta_{m_1'\, m_1+1}\, \delta_{m_2'\, m_2}$

$\to \delta_{m_1'\, m_1}\, \delta_{m_2'\, m_2+1}$

여기에 양 변에 $\langle j_1 j_2 ; m_1'\ m_2'|$ 를 곱.

$$\sqrt{(j-m)(j+m+1)}\,\langle j_1 j_2 ; m_1'\ m_2' | j_1 j_2 ; j\, m+1\rangle = \sqrt{(j_1-m_1)(j_1+m_1+1)}\,\langle j_1 j_2 ; m_1'-1\ m_2' | j_1 j_2 ; j\, m\rangle$$
$$+ \sqrt{(j_2-m_2)(j_2+m_2+1)}\,\langle j_1 j_2 ; m_1'\ m_2'-1 | j_1 j_2 ; j\, m\rangle$$

# CG series

2개의 다른 회전에 대한 rotation matrix의 곱을 1개의 회전에 대한 rotation matrix의 합으로 나타낸다.

유도법: $\langle j_1 j_2\ m_1 m_2 | D(R) | j_1 j_2\ m_1' m_2' \rangle$를 두 가지 방법으로 나타내기

아직 납득이...

① $j_1$과 $j_2$ 따로 돌리기, $D(R) = D^{j_1}(R) + D^{j_2}(R)$

$$\langle j_1 j_2\ m_1 m_2 | D(R) | j_1 j_2\ m_1' m_2' \rangle = \langle j_1\ m_1 | D^{j_1}(R) | j_1\ m_1' \rangle \langle j_2\ m_2 | D^{j_2}(R) | j_2\ m_2' \rangle$$

$$= D^{j_1}_{m_1 m_1'} D^{j_2}_{m_2 m_2'}$$

② 통째로 돌리기. 이때,

$|j_1 j_2 | j\ m\rangle$과 $|j_1 j_2 | j'\ m'\rangle$을 $\mathbb{1}$ 형태로 삽입. D가 $|j_1 j_2\ ;j\ m\rangle$으로 갈 바지게

$$\mathbb{1} = \sum_{j'} \sum_{m'} |j_1 j_2\ ; j' m'\rangle\langle j_1 j_2 ; j' m'|$$

$$\langle j_1 j_2\ m_1 m_2 | D(R) | j_1 j_2\ m_1' m_2' \rangle$$

$$\mathbb{1} = \sum_j \sum_m |j_1 j_2 ; j m\rangle\langle j_1 j_2 ; j m|$$

$$= \sum_j \sum_m \sum_{j'} \sum_{m'} \langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m\rangle \langle j_1 j_2 ; j' m' | j_1 j_2 ; m_1' m_2' \rangle$$

$$\underbrace{\langle j_1 j_2 ; j m | D(R) | j_1 j_2\ j' m' \rangle}$$

$$\delta_{jj'}\ D^{(j)}_{mm'}(R)$$

# Tensor operators

$$T_q^k = Y_{l=k}^{m=q}(\hat{n})$$

T에 회전연산자 취하면 Y의 $(\hat{n})$이 돌아가듯 변환

## Transformation of spherical tensor under rotation

$T_q^k$의 transform은 $Y_{l=k}^{m=q}(V)$에서 V를 회전시킬 때 변화와 똑같다.

먼저, $Y_l^m(\hat{n})$이 어떻게 변환되었는가?

$$Y_l^m(\hat{n}) = \langle \hat{n} | l m \rangle$$

$$Y_l^m(\hat{n}') = \langle D(R)\hat{n} | lm \rangle = \langle \hat{n} | D(-R) | lm \rangle$$

$$= \sum_{m'} \langle \hat{n} | l m' \rangle \langle l m' | D(-R) | lm \rangle$$

$$= \sum_{m'} Y_l^{m'}(\hat{n}) \, D^{(l)}_{m'm}(-R)$$

$$= \sum_{m'=-l}^{l} D^{(l)*}_{mm'}(R) \, Y_l^{m'}(\hat{n})$$

$$D(-R) = D^{-1}(R) = D^{\dagger}(R)$$

$$[D(-R)]_{m'm} = [D^{\dagger}(R)]_{m'm} = [D^{*}(R)]_{mm'}$$

★ 주의.

$$D^{\dagger}(R)\, T_q^k \, D(R) = \sum_{q'=-k}^{k} D^{k*}_{qq'}(R)\, T_{q'}^k \iff D(R)\, T_q^k \, D^{\dagger}(R) = \sum_{q'=-k}^{k} D^{k}_{q'q}(R)\, T_{q'}^k$$

## commutation relation of spherical tensor

위 식에 infinitesimal rotation

$D(R) = 1 - i\frac{\varepsilon}{\hbar}\hat{n}\cdot\vec{J}$ 를 삽입

$$[\hat{n}\cdot\vec{J}, T_q^k] = \sum_{q'=-k}^{k} \langle k q' | \hat{n}\cdot\vec{J} | k q \rangle \, T_{q'}^k$$

$\hat{n} = \hat{z}$ 와 $\hat{n} = \hat{x} \pm i\hat{y}$ 를 대입해 확인

$$[J_z, T_q^k] = \hbar q \, T_q^k$$

$$[J_\pm, T_q^k] = \hbar\sqrt{(k\mp q)(k\pm q+1)}\; T_{q\pm1}^k$$

$(j\mp m)(j\pm m$

마치 $|l=k, m=q\rangle$에 $J_z$와 $J_\pm$를 적용한 것 마냥.

두 개의 spherical tensor의 곱을 더해서 하나의 spherical tensor를 만들기

$$T_q^k = \underbrace{\langle \underbrace{k_1 k_2; m_1=q_1, m_2=q_2}_{X와 Z의 기저} | \underbrace{k_1 k_2; j=k\ m=q}_{T의 기저} \rangle}_{CG\ coefficients} X_{q_1}^{(k_1)} Z_{q_2}^{(k_2)}$$

$Y_1^{0,\pm1}$ 을 $T_{0,\pm1}^1$ 로 변환

$$\hat{n} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \to (U_x, U_y, U_z)$$ 로

($U_x$: $\sin\theta\cos\phi$, $U_y$: $\sin\theta\sin\phi$)

표현하자. $U_z = \frac{z}{r} = \cos\theta$

$$U_\pm = \frac{1}{\sqrt{2}}(U_x \pm iU_y) = \frac{1}{\sqrt{2}}\sin\theta(\cos\phi \pm i\sin\phi)$$

$$= \frac{x \pm iy}{r\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\theta\, e^{\pm i\phi}$$

# Wigner Eckart theorem

핵심 : $T_q^k$ 의 행렬원소 $\langle \alpha' j' m' | T^k | \alpha j m \rangle$ 은

기하학 성분 (CG) $\langle j k ; m q | j k ; j' m' \rangle$ 과

$j_1 = j \quad m_1 = m$
$j_2 = k \quad m_2 = q$
인 두 회전이 있음?

순서 주의!
보통 CG는 $\langle j_1 j_2 ; m_1 m_2 | j_1 j_2 ; j m \rangle$ 이다.

동역학 성분 $\dfrac{\langle \alpha' j' \| T_q^k \| \alpha j \rangle}{\sqrt{2j'-1}}$ 로 나누어짐.

$$\langle \alpha' j' m' | T_q^k | \alpha j m \rangle = \langle j k ; m q | j k ; j' m' \rangle \frac{\langle \alpha' j' \| T^k \| \alpha j \rangle}{\sqrt{2j'+1}}$$

selection rule ① $\rightarrow$ $m' = q + m$. 증명은 $[J_z, T_q^k] = \hbar q T_q^k$ 를 이용.

$\langle \alpha' j' m' | J_z T_q^k - T_q^k J_z - \hbar q T_q^k | \alpha j m \rangle$

$= \hbar (m' - m - q) \langle \alpha' j' m' | T_q^k | \alpha j m \rangle = 0$

$m' = m + q$ 여야만 $\langle \alpha' j' m' | T_q^k | \alpha j m \rangle \neq 0$. 사실 CG에서 바로 나오는 특성.

selection rule ② $\rightarrow$ $j', j, k$ 가 삼각부등식을 만족. $|j-k| \leq j' \leq j+k$

---

증명 : $[J_\pm, T_q^k] = \hbar \sqrt{(k \mp q)(k \pm q + 1)} \; T_{q\pm1}^k = J_\pm T_q^k - T_q^k J_\pm$ 이 두 가지를 대입하고 비교.

$\langle \alpha' j' m' | [J_\pm, T_q^k] | \alpha j m \rangle$

$= \hbar \sqrt{(k \mp q)(k \pm q + 1)} \langle \alpha' j' m' | T_{q\pm1}^k | \alpha j m \rangle$

$= \hbar \sqrt{(j' \pm m')(j' \mp m' + 1)} \langle \alpha' j' m' \mp 1 | T_q^k | \alpha j m \rangle$

$\quad - \hbar \sqrt{(j \mp m)(j \pm m + 1)} \langle \alpha' j' m' | T_q^k | \alpha j m \pm 1 \rangle$

$\sqrt{(j' \pm m')(j' \mp m' + 1)} \langle \alpha' j' m' \mp 1 | T_q^k | \alpha j m \rangle$

$= \sqrt{(k \mp q)(k \pm q + 1)} \langle \alpha' j' m' | T_{q\pm1}^k | \alpha j m \rangle$

$\quad + \sqrt{(j' \pm m')(j' \mp m' + 1)} \langle \alpha' j' m' \mp 1 | T_q^k | \alpha j m \rangle$

Recursion formula of CG coefficient

$\sqrt{(j \mp m)(j \pm m + 1)} \langle j_1 j_2 ; m_1' m_2' | j_1 j_2 ; j \; m+1 \rangle$

$= \sqrt{(j_1 \mp m_1)(j_1 \pm m_1 + 1)} \langle j_1 j_2 ; m_1' - 1 \; m_2' | j_1 j_2 ; j m \rangle$

$\quad + \sqrt{(j_2 \mp m_2)(j_2 \pm m_2 + 1)} \langle j_1 j_2 ; m_1' \; m_2' - 1 | j_1 j_2 ; j m \rangle$

$$\langle \alpha' j' m' | T^k_q | \alpha j m \rangle = \langle j k ; m q | j k ; j' m' \rangle \frac{\langle \alpha' j' \| T^k \| \alpha j \rangle}{\sqrt{2j'+1}}$$

예시 문제

다음 행렬원소가 주어졌을 때: $k=1$

$$\langle j'=1, m'=1 | V_0 | j=1, m=1 \rangle = A = \langle 1 1 ; 1 0 | 1 1 ; 1 1 \rangle \frac{1}{\sqrt{3}} \langle 1 \| V_0 \| 1 \rangle = \frac{1}{\sqrt{6}} \langle 1 \| V \| 1 \rangle$$

아래 두 행렬원소를 $A$로 표현하라.

(a) $\langle j'=1, m'=0 | V_0 | j=1, m=0 \rangle = \langle 1 1 ; 0 0 | 1 1 ; 1 0 \rangle \frac{1}{\sqrt{3}} \langle 1 \| V_0 \| 1 \rangle = \frac{\sqrt{2}}{3} \langle 1 \| V \| 1 \rangle$

(b) $\langle j'=1, m'=1 | V_{+1} | j=1, m=0 \rangle = \langle 1 1 ; 0 1 | 1 1 ; 1 1 \rangle \frac{1}{\sqrt{3}} \langle 1 \| V_1 \| 1 \rangle = \frac{1}{\sqrt{6}} \langle 1 \| V \| 1 \rangle$

$\therefore \langle 1 \| V \| 1 \rangle = \sqrt{6} A$, $\quad$ (a) $= \frac{2}{\sqrt{3}} A$ $\quad$ (b) $= A$

# Symetry, Conservation laws, and Degeneracy  Parity

Unitary Operator : 어떤 transform 을 일으킴

이 transform 에 의해 변형된 Hamiltonian.

→ 샌드위치 하는 방법

$T^\dagger \mathcal{H} T = \mathcal{H}$  transform 해도 그대로?

→ Symmetry 다!

⇕ 동치

$[\mathcal{H}, T] = 0$  $\mathcal{H}$ 와 transformation unitary operator 가 commute.

G 는 T 의 generator

$T = 1 - \frac{i}{\hbar}\varepsilon G$  $[\mathcal{H}, G] = 0$

Generator 가 보존량이다.

| Transform / symmetry | Generator / 보존량 |
|---|---|
| space translation | Linear momentum |
| Rotation | Angular momentum |

Symmetry 가 degeneracy 일으킴.

→ $\mathcal{H}$ 와 T 가 commute 하기 때문

$\mathcal{H} T |a\rangle = T \mathcal{H} |a\rangle$ , $\mathcal{H}|a\rangle = E_a |a\rangle$

$\mathcal{H}|a'\rangle = E_a |a'\rangle$ , $T|a\rangle = |a'\rangle$ 도 똑같은 $E_a$

## Parity → position 샌드위치하면 부호 바뀜

$\pi$

$\pi^\dagger x \pi = -x$ , $x\pi + \pi x = 0$

$\{x, \pi\} = 0$  Anticommute with position

Parity 는 Hermitian 이기도 하다! Observable 임!

$\pi = \pi^{-1} = \pi^\dagger$  셋 모두가 같음

Eigenvalue 가 (아님) → 1 even parity 우함수
→ $-1$ odd parity 기함수

## Momentum 은 odd parity

Space translation operator : $T(dx) = 1 - \frac{i}{\hbar} dx\, p$

$\pi^\dagger T(dx) \pi = T(-dx) = 1 + \frac{i}{\hbar} dx\, p$

$= \pi^\dagger (1 - \frac{i}{\hbar} dx\, p) \pi = 1 - \frac{i}{\hbar} dx\, \pi^\dagger p \pi$

$\pi^\dagger p \pi = -p$ , $\{\pi, p\}$

## Angular momentum 은 even parity

$L = x \times p$ , $\pi^\dagger L \pi = (-x) \times (-p)$

vector
- odd parity : polar vector
- even parity : Axial vector

Scalar
- odd parity : pseudo scalar
- even : real scalar

# Symetry, Conservation laws, and Degeneracy

# Parity

$$\langle x|\pi|\alpha\rangle \begin{cases} = \pm\langle x|\alpha\rangle = \pm\psi(x) \\ \langle -x|\alpha\rangle = \psi(-x) \end{cases}$$

$|\alpha\rangle$가 $\pi$의 eigenstate 인 경우

$$\psi(-x) = \pm\,\psi(x) \qquad \begin{matrix} 1 \to \text{even} \\ -1 \to \text{odd} \end{matrix}$$

---

## Spherical harmonics and parity

$$\pi:\ \cos\theta \longrightarrow -\cos\theta,\quad \phi \to \pi+\phi$$
$$\sin\theta \longrightarrow \sin\theta$$

$$Y_l^m \propto P_l^m(\cos\theta)\,e^{im\phi} \xrightarrow{\pi} P_l^m(-\cos\theta)\,e^{im\phi+\underbrace{im\pi}_{(-1)^m}}$$

$$P_l^m \propto \left(\frac{d}{d\cos\theta}\right)^{l-m}(\sin\theta)^{2l} \xrightarrow{\pi} (-1)^{l-m}\left(\frac{d}{d\cos\theta}\right)^{l-m}(\sin\theta)^{2l}$$

$$\therefore\ Y_l^m \xrightarrow{\pi} (-1)^m(-1)^{l-m}P_l^m(\cos\theta)\,e^{im\phi}$$
$$= (-1)^l\,Y_l^m$$

| |
|---|
| $l$이 even $\longrightarrow$ parity 에도 even |
| $l$이 odd $\longrightarrow$ parity 에도 odd |

---

$[\mathcal{H},\pi]=0$ 이면 $\mathcal{H}$-eigenket은 $\pi$-eigenket이다.

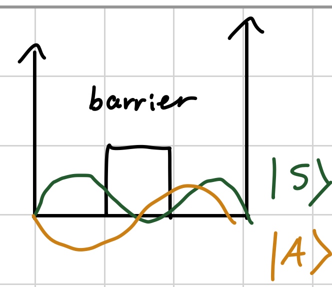
증명 setup: $\mathcal{H}|n\rangle = E_n|n\rangle$

전략 : $\pi\pi=1$ 이라는 특성을 이용해 $|n\rangle$으로 부터 $\pi$-eigenket을 만들고 이것이 $\mathcal{H}$-eigenket 이기도 함을 보이기

$|\alpha\rangle = ((1\pm\pi)|n\rangle$ and, $|n\rangle = \frac{1}{2}(1+\pi)|n\rangle + \frac{1}{2}(1-\pi)|n\rangle$

$\pi|\alpha\rangle = (\pi\pm1)|n\rangle = \pm(1\pm\pi)|n\rangle = \pm|\alpha\rangle$

$$\begin{aligned}\mathcal{H}|\alpha\rangle &= \mathcal{H}|n\rangle \pm \mathcal{H}\pi|n\rangle = E_n|n\rangle \pm E_n\pi|n\rangle \\ &= E_n(1\pm\pi)|n\rangle \\ &= E_n|\alpha\rangle \quad \blacksquare\end{aligned}$$




$\mathcal{H}|A\rangle = E_A|A\rangle$
$\mathcal{H}|S\rangle = E_S|A\rangle$
$E_A - E_S > 0$

$$|R\rangle = \frac{1}{\sqrt{2}}(|S\rangle+|A\rangle),\quad |L\rangle = \frac{1}{\sqrt{2}}(|S\rangle-|A\rangle)$$

$$\begin{aligned}|R,t\rangle &= \frac{1}{\sqrt{2}}\left(e^{-i\frac{E_S}{\hbar}t}|S\rangle + e^{-i\frac{E_A}{\hbar}t}|A\rangle\right) \\ &= \frac{1}{\sqrt{2}}e^{-i\frac{E_S}{\hbar}t}\left(|S\rangle + e^{i\frac{(E_S-E_A)}{\hbar}t}|A\rangle\right)\end{aligned}$$

$|R,t\rangle$와 $|L\rangle$ 가 같아지는 순간이 tunneling.

$e^{-i\frac{E_S-E_A}{\hbar}t} = -1$ 가 되는 순간.

$$\frac{E_S-E_A}{\hbar}t = \pi$$

Tunneling frequency

$\frac{1}{\hbar}(E_S-E_A)$ 두 상태의 에너지 차

| |
|---|
| 모든 상태는 Parity even/odd 로 쪼갤 수 있다. |
| $\begin{aligned}\lvert n\rangle &= \tfrac{1}{2}(1+1+\pi-\pi)\lvert n\rangle \\ &= \underbrace{\tfrac{1}{2}(1+\pi)\lvert n\rangle}_{\text{even}} + \underbrace{\tfrac{1}{2}(1-\pi)\lvert n\rangle}_{\text{odd}}\end{aligned}$ |
| $\begin{cases}\pi(1+\pi) = 1+\pi \\ \pi(1-\pi) = -(1-\pi)\end{cases}$ |

# Symmetry, Conservation laws, and Degeneracy — Lattice Translation

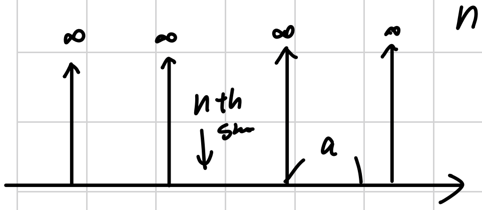
potential 이 공간에 대해 $a$ 만큼의 주기성을 가진다.

이 $a$ 만큼 space translation 하는 operator가

$\tau(a)$ 이건 unitary 지만, Hermitian 은 아니다.

$$x \xrightarrow{\tau(a)} x+a$$

## Dirac Comb




nth site 에 놓여 있는 상태

$|n\rangle$. $n$ 에 상관없이 에너지 같다.

$\mathcal{H}|n\rangle = E_0|n\rangle$ , $\mathcal{H}|m\rangle = E_0|m\rangle$

$\tau(a)|n\rangle = |n+1\rangle \rightarrow |n\rangle$은 $\tau(a)$ 의 eigenstate가 아니다.

$$|\theta\rangle = \sum_n e^{in\theta}|n\rangle$$

이건 $\mathcal{H}$와 $\tau(a)$의 simultaneous eigenket

$\mathcal{H}|\theta\rangle = E_0|\theta\rangle$

$$\tau(a)|\theta\rangle = \sum_n e^{in\theta}|n+1\rangle = \underbrace{e^{-i\theta}}_{\text{eigenvalue}}\sum_{n'} e^{in'\theta}|n'\rangle$$

$n' = n+1$

$|\theta\rangle$의 $\tau(a)$ eigen value는 $e^{-i\theta}$

## Finit periodic barrier

$\langle n+1|\mathcal{H}|n\rangle = -\Delta$ , $\langle n|\mathcal{H}|n\rangle = E_0$

$\mathcal{H}|n\rangle = E_0|n\rangle - \Delta|n+1\rangle - \Delta|n-1\rangle$

$|\theta\rangle$ 는 여전히 $\mathcal{H}$- eigenstate 일까?

$$\mathcal{H}|\theta\rangle = E_0\sum_n e^{in\theta}|n\rangle - \Delta\sum_m e^{im\theta}|m+1\rangle - \Delta\sum_k e^{ik\theta}|k-1\rangle$$

$n=m+1$ $n=k-1$

$$= (E_0 - \Delta e^{-i\theta} - \Delta e^{i\theta})\sum_n e^{in\theta}|n\rangle$$
$$= (E_0 - 2\Delta\cos\theta)|\theta\rangle$$

$$E(\theta) = E_0 - \underbrace{2\Delta}_{\text{energy band width.}}\cos\theta$$

## $|\theta\rangle$의 wave function — Bloch's theorem

$$\langle x|\tau(a)|\theta\rangle \rightarrow e^{-i\theta}\langle x|\theta\rangle = e^{-i\theta}\psi(x)$$
$$\langle x|\tau(a)|\theta\rangle \rightarrow \langle \tau^\dagger(a)x|\theta\rangle = \langle x-a|\theta\rangle = \psi(x-a)$$

$$\therefore \psi(x-a) = e^{-i\theta}\psi(x)$$

병진 이동이 위상을 바꿈

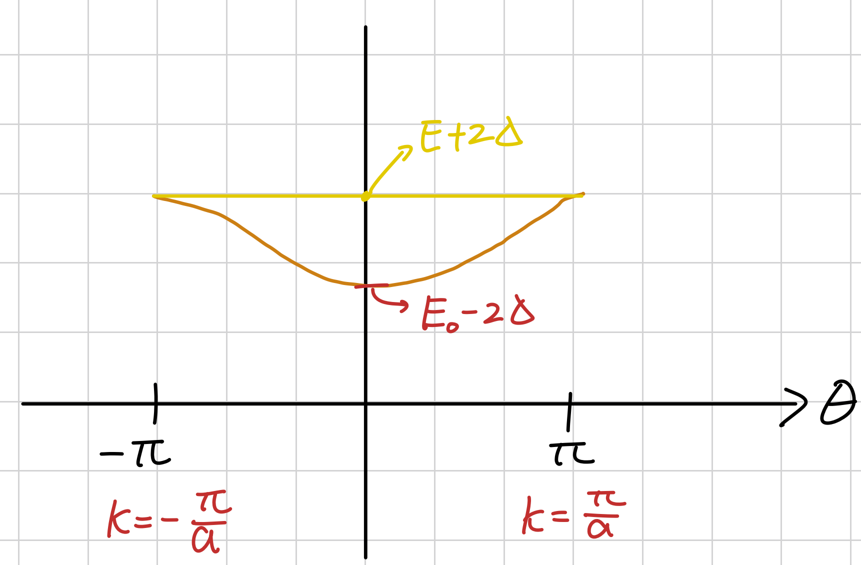
위 방정식을 만족하는 solution.

$$\psi_\theta(x) = e^{i\frac{\theta}{a}x}u(x) = e^{ikx}u(x)$$

$k = \frac{\theta}{a}$ , $u(x\pm a) = u(x)$ $\forall x$

즉, $u$는 $a$ 주기로 periodic한 함수 아무거나

$$\psi_\theta(x) = e^{ik(\theta)x}u(x)$$




1st Brillouin Zone

# Symmetry, Conservation laws, and Degeneracy — Time Reversal

위치는 그대로, 운동 방향만 거꾸로

$x \xrightarrow{\Theta} x \qquad p \xrightarrow{\Theta} -p$

EOM 이 p의 → even power law → TR sym
→ odd power law → TR sym broken

## 슈방이 TR에 대칭이라면, $\psi^*(-t)$도 solution 이다.

$$i\hbar \frac{\partial\psi(t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(t)}{\partial x^2} \xrightarrow{\Theta} -i\hbar\frac{\partial\psi(-t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(-t)}{\partial x^2}$$

$\psi(-t)$는 슈방의 해가 아니다!
대신, 식 전체에 C.C 하면,

$$i\hbar\frac{\partial\psi^*(-t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*(-t)}{\partial x^2}$$

$\psi^*(-t)$ 가 슈방의 해다.

## Antiunitary Operator

꼭 내적을 보존할 필요 없이, $|\langle\beta'|\alpha'\rangle|^2 = |\langle\beta|\alpha\rangle|^2$ 이기만 해도 되잖아?

Definition: $|\beta'\rangle = \Theta|\beta\rangle$, $|\alpha'\rangle = \Theta|\alpha\rangle$

$$\langle\alpha'|\beta'\rangle = \langle\alpha|\beta\rangle^* = \langle\beta|\alpha\rangle$$

$$\Theta = UK$$

U와 K는 교환 안함. 주의

complex conjugate operator

K: coefficient 를 C.C.
그러나 basis 는 가만히.

$|\alpha\rangle = \sum_n \langle n|\alpha\rangle|n\rangle$, $|n\rangle$을 basis 삼으면,

$UK|\alpha\rangle = |\alpha'\rangle = \sum_n \langle n|\alpha\rangle^* U|n\rangle = \sum_n \langle\alpha|n\rangle U|n\rangle$

$\langle\beta'| = \sum_n \langle n|\beta\rangle\langle n|U^\dagger$

$\langle\beta|\alpha\rangle = \sum_n \langle\beta|n\rangle\langle n|\alpha\rangle$

$\langle\alpha|\beta\rangle = \sum_n a_n^* b_n = \sum_n \langle\alpha|n\rangle\langle n|\beta\rangle$

$= \sum_n \langle n|\beta\rangle\langle n|U^\dagger U|n\rangle\langle\alpha|n\rangle$

$= \langle\beta'|\alpha'\rangle$

Time reversal operator 와 time-evolution 둘의 관계는?

System이 time reversal symmetry 를 가지면,

① TR 한 다음 $\delta t$ 진화시키기
② $-\delta t$ 진화한 다음 TR 하기
→ 둘이 같아야.

$\delta t$ 가 작은 infinitesimal operator 일 때,

$$\underbrace{\left(1-\frac{i}{\hbar}\mathcal{H}\delta t\right)\Theta}_{①\ 방식} = \underbrace{\Theta\left(1+\frac{i}{\hbar}\mathcal{H}\delta t\right)}_{②\ 방식}$$

$-\mathcal{H}\Theta = \Theta\mathcal{H}$ ... 라고 하면 안 되지!

$-i\mathcal{H}\Theta = \Theta i\mathcal{H}$ 라고 해야지!

case 1) $\Theta$가 unitary: $\mathcal{H}|n\rangle = E_n|n\rangle$

$\mathcal{H}\Theta|n\rangle = \mathcal{H}|\tilde{n}\rangle = -\Theta\mathcal{H}|n\rangle = -E_n|\tilde{n}\rangle$

time-reversed energy eigenstate 가 음의 에너지 이상함.

case 2) $\Theta$ 가 antiunitary

$-i\mathcal{H}UK = UKi\mathcal{H} = -iUK\mathcal{H}$

$\therefore \mathcal{H}\Theta = \Theta\mathcal{H}$, $[\mathcal{H}, \Theta] = 0$.

이러면 tR-state도 같은 에너지 가짐.

$$\langle\beta|A|\alpha\rangle = \langle\tilde{\alpha}|\Theta A\Theta^{-1}|\tilde{\beta}\rangle$$

순서가 뒤집혔어

유도법... 몰라도 되겠지?
$A|\beta\rangle = |\gamma\rangle$ 로 둠.
$\langle\gamma|\alpha\rangle = \langle\tilde{\alpha}|\tilde{\gamma}\rangle$

TR 했을 때 A의 expectation value 부호가 바뀌나?

$$\Theta A\Theta^{-1} = \begin{cases} A : \text{TR even operator} & \langle\alpha|A|\alpha\rangle = \langle\tilde{\alpha}|A|\tilde{\alpha}\rangle \\ -A : \text{TR odd operator} & \langle\alpha|A|\alpha\rangle = -\langle\tilde{\alpha}|A|\tilde{\alpha}\rangle \end{cases}$$

expectation value

$$\langle\alpha|A|\alpha\rangle \xrightarrow{TR} \langle\tilde{\alpha}|\Theta A\Theta^{-1}|\tilde{\alpha}\rangle$$

# Symmetry, Conservation laws, and Degeneracy — Time Reversal

## Time reversal 에도 commutation relation은 보존된다!

→ 당연히 그래야 한다.

예시) $[x,p] = i\hbar$ 에 적용. 아무 ket $|\rangle$에 대해.

$\Theta[x,p]|\rangle = \Theta i\hbar|\rangle$ , $\Theta[x,p]\Theta^{-1}\Theta|\rangle = -i\hbar\Theta|\rangle$

$[\Theta x\Theta^{-1}, \Theta p\Theta^{-1}]\Theta|\rangle = -i\hbar\Theta|\rangle$

$[x,-p] = -[x,p]$

## TR된 state는 position wave function에 complex conjugate이 이뤄진다. $\psi(x) \to \psi^*(x)$

$\Theta|x\rangle = |x\rangle$ 이기 때문이다.

$\psi(x) = \langle x|\alpha\rangle$

$\tilde{\psi}(x) = \langle x|\Theta|\alpha\rangle = ?$

$|\alpha\rangle = \int \psi(x)|x\rangle dx$

$\Theta|\alpha\rangle = \int \Theta\psi(x)|x\rangle dx = \int \psi^*(x)|x\rangle dx$

## Position-wave function은 $\phi(p) \to \phi^*(-p)$

$\Theta|p\rangle = |-p\rangle$ 이기 때문이다.

$\Theta|\alpha\rangle = \int_{-\infty}^{\infty} \Theta\phi(p)|p\rangle dp = \int_{-\infty}^{\infty} \phi^*(p)|-p\rangle dp = -\int_{\infty}^{-\infty} \phi^*(-p')|p'\rangle dp'$

$= \int_{-\infty}^{\infty} \phi^*(-p')|p'\rangle dp'$

## Probability current는 TR에 뒤집어진다. $J \xrightarrow{\Theta} -J$

$J = \frac{\hbar}{m} Im[\psi^*\nabla\psi] \longrightarrow \frac{\hbar}{m} Im[\psi\nabla\psi^*]$

$= \frac{\hbar}{m} Im[(\psi^*\nabla\psi)^*]$

$= -\frac{\hbar}{m} Im[\psi^*\nabla\psi] = -J$

앞서 본, $\psi(x) \to \psi^*(x)$ 에 의한 결과이다.

## Energy eigenstate의 position wave function $\psi(x)$는 실수이다.

$|n\rangle$과 $\Theta|n\rangle$의 energy eigen value가 같음.

$|n\rangle$과 $\Theta|n\rangle$의 position wave function이 같아야 함. $\psi_n(x) = \psi_n^*(x)$

이러면 $Im[\psi^*\nabla\psi] = 0$ 이다.

probability current가 0.

Time-evolution에 변화가 없는 stationary state 라는 뜻!

# Mixed state

density operator : $\rho = \sum_i w_i |\psi_i\rangle\langle\psi_i|$

$w_i$는 $|\psi_i\rangle$로 관측될 확률.

기댓값 : $\langle Q\rangle = \sum_i w_i \langle\psi_i|Q|\psi_i\rangle$

$= Tr(Q\rho)$

$= \sum_j \langle j|Q\rho|j\rangle$

$= \sum_j \sum_i w_i \langle b_j|Q \underbrace{|\psi_i\rangle\langle\psi_i|}_{1} b_j\rangle$

$= \sum_j w_i \langle b_j|Q|b_j\rangle$

$\langle Q\rangle = Tr(Q\rho)$

pure state : $\rho^2 = \rho$ $\quad Tr(\rho^2) = 1$

$\rho^2 \neq \rho$ $\quad Tr(\rho^2) < 1.$

$S = -k\, Tr(\rho \ln \rho)$ $\qquad \rho = \sum_i p_i |a_i\rangle\langle a_i|$

$S = -k\, Tr(p_i \ln p_i)$

# Schwinger Oscillator model

## 기존 연산자와 $a_+^\dagger, a_+, a_-^\dagger, a_-$ 의 관계

$$N = a^\dagger a \quad , \quad N+1 = a a^\dagger$$

$$J_+ = \hbar a_+^\dagger a_- \quad , J_- = \hbar a_+ a_-^\dagger$$

$$J_z = \frac{\hbar}{2}(N_+ - N_-) = \frac{\hbar}{2}\left(a_+^\dagger a_+ - a_-^\dagger a_-\right)$$

$$J_x^2 + J_y^2 = \frac{J_+J_- + J_-J_+}{2} = \frac{\hbar^2}{2}\left(\underbrace{a_+^\dagger a_- a_+ a_-^\dagger}_{a_+^\dagger a_+ a_- a_-^\dagger} + \underbrace{a_+ a_-^\dagger a_+^\dagger a_-}_{a_+ a_+^\dagger a_-^\dagger a_-}\right)$$

$$N_+(N_- + 1) + N_-(N_+ + 1) = 2N_+N_- + N_+ + N_-$$

$$J_z^2 = \frac{\hbar^2}{4}\left(N_+^2 + N_-^2 - 2N_+N_-\right)$$

$$J^2 = \frac{\hbar^2}{4}\left(N_+^2 + N_-^2 + 2N_+N_- + 2N_+ + 2N_-\right)$$

$$= \frac{\hbar^2}{2}\left(\frac{N_+^2 + N_-^2}{2} + N_+N_- + N_+ + N_-\right)$$

$$= \frac{\hbar}{2}(N_+ + N_-)\left(\frac{N_+ + N_-}{2} + 1\right)$$

$$= \hbar \frac{N}{2}\left(\frac{N}{2} + 1\right)$$

$\frac{N}{2}$ 이 j다.

$$m = \frac{1}{2}(n_+ - n_-)$$

$$j = \frac{1}{2}(n_+ + n_-)$$

$$(x+y)^n = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} x^k y^{n-k}$$