2026 Classical Mechanics Final term.

1. Based on the following effective action

$$S = \int \left((\boldsymbol{p} + e\boldsymbol{A}) \cdot \frac{d\boldsymbol{x}}{dt} - (|\boldsymbol{p}| + e\phi) - \boldsymbol{a} \cdot \frac{d\boldsymbol{p}}{dt} \right) dt \qquad \nabla_p \times \boldsymbol{a} = \Theta \equiv \frac{\hat{\boldsymbol{p}}}{2|\boldsymbol{p}|^2} \qquad \hat{\boldsymbol{p}} = \boldsymbol{p}/|\boldsymbol{p}|$$

, where a_μ is a momentum-space Berry connection, derive the Hamiltonian equation of motion and solve them to find the below expression

$$\begin{cases} \mathfrak{m} \frac{d\boldsymbol{x}}{dt} = \hat{\boldsymbol{p}} + e\boldsymbol{E} \times \Theta + (\Theta \cdot \hat{\boldsymbol{p}}) e\boldsymbol{B}, \\ \mathfrak{m} \frac{d\boldsymbol{p}}{dt} = e\boldsymbol{E} + e\hat{\boldsymbol{p}} \times \boldsymbol{B} + e^2 (\boldsymbol{E} \cdot \boldsymbol{B}) \Theta, \end{cases} \qquad \mathfrak{m} = 1 + e\Theta \cdot \boldsymbol{B}$$

Here, m may be regarded as an effective mass for this particle. Can you imagine a UV complete version of this effective action? In other words, figure out where this effective action comes from. Maybe, google or AI with "chiral fermions". It is important to notice the momentum-linear dispersion, which implies that Poincare invariance = Lorentz invariance + Translational invariance will play a key role. Can you figure out the Lorentz symmetry in this particle dynamics (optional)?

(50 points)

2. Revisit this problem based on the symplectic geometry perspectives. Can you find the corresponding symplectic two form and the symplectic vector field? Let me give you the answer.

$V^7 = T(\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}$ = $(\boldsymbol{x}, \boldsymbol{p}, t)$ 6-dimensional phase space + 1 time dimension

$$\sigma = \omega - dh \wedge dt,$$

$$\omega = \omega_0 + \frac{e}{2} \epsilon_{ijk} B^i dx^j \wedge dx^k, \qquad i_{X_H}\omega = dH \text{ or the following expression}$$

$$\omega_0 = dp_i \wedge dx^i - \frac{s}{2|\boldsymbol{p}|^3} \epsilon^{ijk} p_i dp_j \wedge dp_k, \qquad \omega_{\alpha\beta} \dot{\xi}^\beta = \partial_\alpha h, \quad \text{where} \quad \omega_{\alpha\beta} = \partial_\alpha u_\beta - \partial_\beta u_\alpha.$$

$$h = |\boldsymbol{p}| + e\phi,$$

Clarify the symplectic structure. Show that this symplectic two form is non-degenerate, i.e., given by $\det(\omega_{\alpha\beta}) \neq 0$

(50 points)

3. Consider a spinning particle to follow relativity. To describe the spinning particle, we consider a 9-dimensional phase space, given by

$$V^9 = \{R, I, J \in \mathbb{R}^{3,1} | I_\mu I^\mu = J_\mu J^\mu = 0, I_\mu J^\mu = -1\} \quad .$$

Here, $\mathrm{I}_\mu \,\&\, J_\mu$ are two independent null vectors, given above. To figure out that this is the 9-dimensional phase space, we introduce $\mathrm{P}_\mu \,\&\, S_{\mu\nu}$ to replace $\mathrm{I}_\mu \,\&\, J_\mu$ as follows: $\mathrm{I}_\mu \to P_\mu \,\&\, \mathrm{S}_{\mu\nu} = -s\epsilon_{\mu\nu\rho\sigma}P^\rho J^\sigma$. Then, we obtain

$$V^9 = \left\{R, P \in \mathbb{R}^{3,1}, S \in \mathfrak{o}(3,1) | P_\mu P^\mu = 0, \right.$$
$$\left. S_{\mu\nu}P^\nu = 0, \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = s^2 \right\},$$

It is straightforward to check out that $\mathrm{P}_\mu \,\&\, S_{\mu\nu}$ satisfy these constraints.

Now, we introduce the following closed two form $\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu_\lambda \wedge S^\lambda_\rho dS^\rho_\mu.$

Here, we do not consider electromagnetic fields. Identify the Hamiltonian vector field and obtain the following Hamiltonian equation of motion,

$$\begin{cases} P_\mu \dot{R}^\mu = 0, \\ \dot{P}^\mu = 0, \\ \dot{S}^{\mu\nu} = P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu, \end{cases}$$

It is quite surprising to realize that this symplectic two form is identical to that of Q. 2 if electromagnetic fields are neglected, i.e., ω_0 in Q. 2. In other words, the symplectic two form of Q. 2 without electromagnetic fields is a symplectic reduction of the above symplectic two form. Can you figure it out? See below.

To obtain down-to-earth expressions, we put R = (**r** , t), where **r** and t are the position and time coordinates in a chosen Lorentz frame. The two null-vectors are in turn P = (**p** , |**p**|) and J = (**q** , - |**q**|), where **p** and **q** are two (necessarily nonzero) 3-vectors which satisfy **p** · **q** + |**p**| |**q**| = 1.

In these terms we have

$$S_{ij} = \epsilon_{ijk} s^k, \qquad \boldsymbol{s} = s(\boldsymbol{p}|\boldsymbol{q}| + \boldsymbol{q}|\boldsymbol{p}|),$$
$$S_{j4} = s(\boldsymbol{p} \times \boldsymbol{q})_j = (\hat{\boldsymbol{p}} \times \boldsymbol{s})_j.$$

Based on this coordinate representation, find the moment map that reduces the symplectic two form in 9-dimensional phase space to that in 6-dimensional one.

(100 points)

4. Now, we introduce a non-minimal coupling of electromagnetic fields into this spin 1/2 chiral fermion. Starting from the closed two form,

$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu_\lambda \wedge S^\lambda_\rho dS^\rho_\mu + \frac{1}{2} eF_{\mu\nu} dR^\mu \wedge dR^\nu$$

derive the corresponding equations of motion as follows:

$$\begin{cases} \dot{R}^\mu = P^\mu + \frac{S^{\mu\nu}F_{\nu\rho}P^\rho}{\frac{1}{2}S\cdot F}, \\ \dot{P}^\mu = -eF^\mu_\nu \dot{R}^\nu, \\ \dot{S}^{\mu\nu} = P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu. \end{cases}$$

Unfortunately, the above closed two form is not completely satisfactory. We have to introduce the mass square constraint in the following way: $P_\mu P^\mu = -\frac{eg}{2}S\cdot F.$

Hence we introduce the novel evolution space

$$\tilde{V}^9 = \Big\{ P, R \in \mathbb{R}^{3,1}, S \in \mathfrak{o}(3,1) | P_\mu P^\mu = -\frac{eg}{2} S\cdot F, \\ S_{\mu\nu}P^\nu = 0, \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = s^2 \Big\}, \tag{5.11}$$

endowed with the closed two-form,

$$\sigma = -dP_\mu \wedge dR^\mu - \frac{1}{2s^2} dS^\mu_\lambda \wedge S^\lambda_\rho dS^\rho_\mu + \frac{1}{2} eF_{\mu\nu} dR^\mu \wedge dR^\nu. \tag{5.12}$$

As a result, we find

$$\begin{cases} \dot{R}^\mu = P^\mu - \frac{1}{(g+1)}\frac{1}{S_{\alpha\beta}F^{\alpha\beta}}[(g-2)S^{\mu\nu}F_{\nu\rho}P^\rho - gS^{\mu\nu}\partial_\nu F_{\rho\sigma}S^{\rho\sigma}], \\ \dot{P}^\mu = -eF^\mu_\nu \dot{R}^\nu - \frac{eg}{4}\partial^\mu F_{\rho\sigma}S^{\rho\sigma}, \\ \dot{S}^{\mu\nu} = P^\mu \dot{R}^\nu - P^\nu \dot{R}^\mu + \frac{eg}{2}[S^\mu_\rho F^{\rho\nu} - S^\nu_\rho F^{\rho\mu}]. \end{cases}$$

These equations constitute the zero-rest-mass counterparts of the celebrated Bargmann-Michel-Telegdi equations for massive relativistic particles.

For the massive case, one can show that these types of equations can be derived from the following Lagrangian:

$$L = -\tfrac{1}{2}m(\dot{x}_\mu + \lambda^\nu S_{\mu\nu})(\dot{x}^\mu + \lambda_\nu S^{\mu\nu}) - eA_\mu \dot{x}^\mu \\ + \tfrac{1}{2}(\psi_A \dot{\varphi}^A - \dot{\psi}_A \varphi^A) - \tfrac{1}{2}kF_{\mu\nu}S^{\mu\nu},$$

where

$$S_{\mu\nu} = \psi_A (\Sigma_{\mu\nu})^A_B \varphi^B . \tag{2}$$

The dynamical variables are the position of the particle $x^\mu(\theta)$, a real vector φ^A belonging to a finite-dimensional representation of the Lorentz-group, whose infinitesimal generators are the quantities $(\Sigma_{\mu\nu})^A_B$, a vector ψ_A belonging to the conjugate representation, and the lagrangian multipliers λ^μ, which will play an important role in the following. The constant m represents the mass of the particle and k is given by

$$k = eg/2m , \tag{3}$$

where g represents the gyromagnetic factor.

Starting from this effective Lagrangian, derive

S^2 is a conserved quantity. We note that in the weak-field approximation, all the components of the four-vector η_μ are very small, and therefore they can be neglected, so we get the BMT [5] equations

$$\ddot{x}_\mu = (e/m)F_{\mu\nu}\dot{x}^\nu , \\ \dot{S}_\rho = g(e/2m)F_{\rho\mu}S^\mu + (g-2)(e/2m)F_{\mu\nu}S^\mu \dot{x}^\nu \dot{x}_\rho . \tag{21}$$

(100 points)