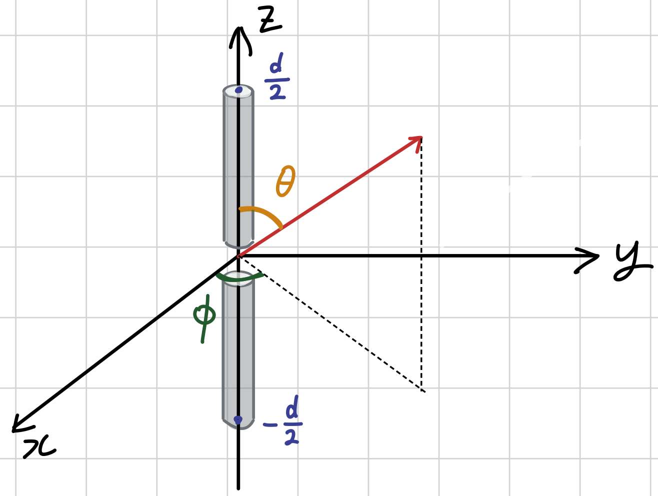
1. Radiation from a center-fed linear antenna.



$$\mathbf{J}(\mathbf{x}) = I\sin\left(\frac{kd}{2} - k|z|\right)\delta(x)\,\delta(y)\,\hat{z}$$

$$|z| < \frac{d}{2}$$

a) Vector potential 을 찾아라.

$\mathbf{J}(\mathbf{x},t) = e^{-i\omega t}\mathbf{J}(\mathbf{x})$ 로 current density 가 oscillate 할 때,

retarded Green's function, $G(\mathbf{x},\mathbf{x}',t,t') = \frac{\mu_0}{4\pi}\cdot\frac{\delta(|\mathbf{x}-\mathbf{x}'| - c(t-t'))}{|\mathbf{x}-\mathbf{x}'|}$

$\mathbf{A}(\mathbf{x},t)$ 를 구하면,

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi}\iint d\mathbf{x}'\,dt'\,\frac{\mathbf{J}(\mathbf{x}',t')}{|\mathbf{x}-\mathbf{x}'|}\,\delta(|\mathbf{x}-\mathbf{x}'| - c(t-t'))$$

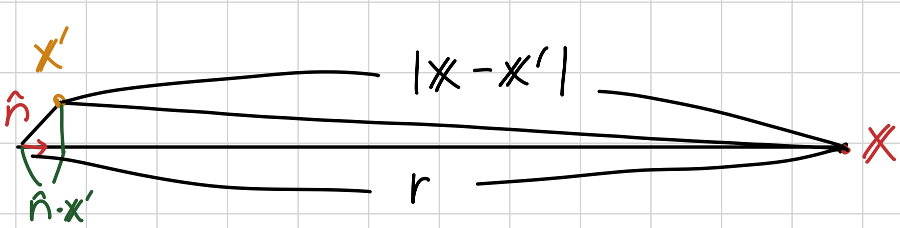
$$= \frac{\mu_0}{4\pi}\iint d\mathbf{x}'\,dt'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,\delta(|\mathbf{x}-\mathbf{x}'| - c(t-t'))\,e^{-i\omega t'}$$

$$= \frac{\mu_0}{4\pi}\int d\mathbf{x}'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,e^{i\frac{\omega}{c}|\mathbf{x}-\mathbf{x}'|}\cdot e^{-i\omega t}$$

$\mathbf{A}(\mathbf{x},t) = \mathbf{A}(\mathbf{x})\,e^{-i\omega t}$ 로 시간 성분과 공간 성분을 분리, 그리고 $c = k\omega$

$$\mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d\mathbf{x}'\,\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,e^{ik|\mathbf{x}-\mathbf{x}'|}$$

이때, $\mathbf{x}$ 에 비해 적분 내부항이 0 이 아닌 $\mathbf{x}'$ 의 범위가 작다는 근사.
(far field, small source) $\mathbf{x}$ 방향의 unit vector 를 $\hat{n}$



$$\boxed{|\mathbf{x}-\mathbf{x}'| \simeq r - \hat{n}\cdot\mathbf{x}'}$$

근사를 대입하면,

$$\mathbb{A}(x) = \frac{\mu_0}{4\pi}\frac{e^{ikr}}{r}\int dx' \, \mathbb{J}(x') \, e^{-ik\hat{n}\cdot x'}$$

문제속 상황의 $\mathbb{J}(x')$ 를 대입. dz 만 적분하면 된다.

$$\mathbb{A}(x) = \hat{z}\frac{\mu_0}{4\pi}\frac{Ie^{ikr}}{r}\int_{-\frac{d}{2}}^{\frac{d}{2}} \sin\left(\frac{kd}{2} - k|z|\right) e^{-ikz\cos\theta} \, dz$$

적분 범위가 0 에 대칭, 기함수는 적분하면 0 이 되니 우함수만 남긴다.

$$e^{ikz\cos\theta} = \cos(kz\cos\theta) - i\sin(kz\cos\theta),$$ $\cos(kz\cos\theta)$ 만 남긴다.

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} \sin\left(\frac{kd}{2} - k|z|\right)\cos(kz\cos\theta)\,dz = 2\int_0^{\frac{d}{2}} \sin\left(\frac{kd}{2} - kz\right)\cos(kz\cos\theta)\,dz$$

$\phi = kz$ 치환 $$\int_0^{\frac{d}{2}} \sin\left(\frac{kd}{2} - kz\right)\cos(kz\cos\theta)\,dz = \int_0^{\frac{kd}{2}} \sin\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta)\,dz$$

$$\int \sin\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta)\,d\phi = \cos\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta) + \int \cos\left(\frac{kd}{2} - \phi\right)\cos\theta\sin(\phi\cos\theta)\,d\phi$$

$$= \cos\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta) - \sin\left(\frac{kd}{2} - \phi\right)\cos\theta\sin(\phi\cos\theta) + \int \sin\left(\frac{kd}{2} - \phi\right)\cos^2\theta\cos(\phi\cos\theta)\,d\phi$$

$$\int \sin\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta)\,d\phi = \frac{1}{\sin^2\theta}\left[\cos\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta) - \sin\left(\frac{kd}{2} - \phi\right)\cos\theta\sin(\phi\cos\theta)\right]$$

$\phi = 0$ 대입, $\dfrac{\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}$ $\phi = \frac{kd}{2}$ 대입, $\dfrac{\cos\left(\frac{kd}{2}\cos\theta\right)}{\sin^2\theta}$ $$\therefore \int_0^{\frac{kd}{2}} \sin\left(\frac{kd}{2} - \phi\right)\cos(\phi\cos\theta)\,dz = \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}$$

적분한 값을 대입하면 벡터 포텐셜은,

$$\boxed{\mathbb{A}(x) = \hat{z}\frac{\mu_0}{4\pi}\frac{2Ie^{ikr}}{kr}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]}$$

$$\mathbb{A}(x) = \hat{z}\,\frac{\mu_0}{4\pi}\cdot\frac{2I}{k}\cdot\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

b) $\mathbb{H}$와 $\mathbb{E}$ field를 구하라.

$$\mathbb{H} = \frac{1}{\mu_0}\nabla\times\mathbb{A}\ , \quad \mathbb{E} = \frac{i}{k}\sqrt{\frac{\mu_0}{\varepsilon_0}}\,\nabla\times\mathbb{H}$$

$$\frac{\partial}{\partial r}\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] = \left(ik-\frac{1}{r}\right)\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

$$\frac{\partial A_z}{\partial r} = \frac{\mu_0}{4\pi}\cdot\frac{2I}{k}\left(ik-\frac{1}{r}\right)\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

$$\frac{\partial}{\partial\theta}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right] = \frac{-2\cos\theta\cos\left(\frac{kd}{2}\cos\theta\right)+2\cos\theta\cos\left(\frac{kd}{2}\right)}{\sin^3\theta} + \frac{\frac{kd}{2}\sin\left(\frac{kd}{2}\cos\theta\right)}{\sin\theta}$$

$$\frac{\partial A_z}{\partial\theta} = \frac{\mu_0}{4\pi}\cdot\frac{2I}{k}\cdot\frac{e^{ikr}}{r}\left(\frac{-2\cos\theta\cos\left(\frac{kd}{2}\cos\theta\right)+2\cos\theta\cos\left(\frac{kd}{2}\right)}{\sin^3\theta} + \frac{\frac{kd}{2}\sin\left(\frac{kd}{2}\cos\theta\right)}{\sin\theta}\right)$$

$$\nabla\times(A_z\hat{z}) = (\nabla A_z)\times\hat{z}$$

$$\nabla A_z = \frac{\partial A_z}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial A_z}{\partial\theta}\hat{\theta}$$

$$(\nabla A_z)\times\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$\nabla\times\mathbb{A} = \left(-\frac{\partial A_z}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial A_z}{\partial\theta}\cos\theta\right)\hat{\phi}$$

$$\mathbb{H} = \frac{Ie^{ikr}}{2\pi k}\cdot\frac{1}{r}\left[\frac{ik-\frac{1}{r}}{\sin\theta}\left\{\cos\left(\frac{kd}{2}\right)-\cos\left(\frac{kd}{2}\cos\theta\right)\right\} + \frac{+2\cos^2\theta\cos\left(\frac{kd}{2}\cos\theta\right)-2\cos^2\theta\cos\left(\frac{kd}{2}\right)}{r\sin^3\theta} - \frac{\frac{kd}{2}\sin\left(\frac{kd}{2}\cos\theta\right)\cos\theta}{r\sin\theta}\right]$$

정신 나갈 것 같이 생겼다. 이렇게 계산하는 것은 너무 힘드니까,

far-field approximation $\nabla \longrightarrow ik\hat{r}$ 을 이용 하자

$$\mathbb{H} = ik\,\hat{r}\times\frac{\mathbb{A}}{\mu_0}\ , \quad \mathbb{E} = -iZ_0k\,\hat{r}\times\left(\hat{r}\times\frac{\mathbb{A}}{\mu_0}\right) = -i\frac{Z_0k}{\mu_0}\left\{\hat{r}(\hat{r}\cdot\mathbb{A})-\mathbb{A}\right\} = i\frac{Z_0k}{\mu_0}\left(\hat{z}-\hat{r}\cos\theta\right)A_z$$

$$\hat{r}\times\hat{z} = -\hat{x}\sin\theta\sin\phi + \hat{y}\sin\theta\cos\phi\ , \qquad \hat{r}(\hat{r}\cdot\hat{z}) = \hat{r}\cos\theta$$

$$\mathbb{H} = i\frac{I}{2\pi}\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right](\hat{r}\times\hat{z}) = i\frac{I}{2\pi}\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]\left(-\hat{x}\sin\theta\sin\phi+\hat{y}\sin\theta\cos\phi\right)$$

$$\mathbb{E} = i\frac{Z_0 k}{\mu_0}(\hat{z}-\hat{r}\cos\theta)\frac{\mu_0}{4\pi}\cdot\frac{2I}{k}\cdot\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]$$

$$\mathbb{E} = i\frac{Z_0 I}{2\pi}\frac{e^{ikr}}{r}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right](\hat{z}-\hat{r}\cos\theta)$$

c) Time averaged power per solid angle을 계산하라.

$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{2}\mathrm{Re}\left[r^2\hat{r}\cdot(\mathbb{E}\times\mathbb{H}^*)\right]$ 를 계산해 보자.

$$\hat{r}\cdot(\mathbb{E}\times\mathbb{H}^*) = Z_0\left(\frac{I}{2\pi}\right)^2\frac{1}{r^2}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right]^2\hat{r}\cdot\{\underbrace{(\hat{z}-\hat{r}\cos\theta)}_{\hat{r}\times(\hat{r}\times\hat{z})}\times(\hat{r}\times\hat{z})\}$$

$$\hat{r}\cdot[\{\hat{r}\times\underbrace{(\hat{r}\times\hat{z})}_{\vec{a}}\}\times\underbrace{(\hat{r}\times\hat{z})}_{\vec{a}}] = |\vec{a}|^2 - |\vec{a}\cdot\hat{r}|^2 = |\hat{r}\times\hat{z}|^2 - |(\hat{r}\times\hat{z})\cdot\hat{r}|^2 = |\hat{r}\times\hat{z}|^2 = \sin^2\theta$$

$$\mathrm{Re}\left[r^2\hat{r}\cdot(\mathbb{E}\times\mathbb{H}^*)\right] = Z_0\frac{I^2}{4\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin^2\theta}\right|^2\sin^2\theta$$

$$= Z_0\frac{I^2}{4\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2$$

마지막으로 여기에 $\frac{1}{2}$을 곱하면!

$$\boxed{\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\left|\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\sin\theta}\right|^2}$$

d) Long wavelength limit 에서 $\left\langle \frac{dP}{d\Omega} \right\rangle \propto (kd)^2 \sin^2\theta$ 꼴로 dipole radiated power 로 reduce 됨을 보여라.

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2} \left| \frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \right|^2$$

$kd \to 0$ 일 때 삼각함수 부분이 어떻게 변형 되는가?

무작정 테일러 전개, $\psi = \frac{kd}{2}$

$$\cos\left(\frac{kd}{2}\cos\theta\right) = \cos(\psi\cos\theta) \simeq 1 - \frac{1}{2}(\psi\cos\theta)^2 + \cdots$$

$$\cos\left(\frac{kd}{2}\right) \simeq 1 - \frac{1}{2}\psi^2 + \cdots$$

$$\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right) = \frac{1}{2}\psi^2(1-\cos^2\theta) = \frac{1}{2}\psi^2\sin^2\theta$$

$$\frac{dP}{d\Omega} \simeq \frac{Z_0 I^2}{8\pi^2}\left|\frac{1}{2}\left(\frac{kd}{2}\right)^2 \sin\theta\right|^2 = \frac{Z_0 I^2}{8^3\pi^2}(kd)^2|\sin\theta|^2 = \frac{Z_0 I^2}{512\pi^2}(kd)^2|\sin\theta|^2$$

문제에서는 분모에 8128이 나온다 했는데, 그럴리가 없다.

e) $kd = \pi$ 일 때, $kd = 2\pi$ 일 때의 특수 상황에서 $\frac{dP}{d\Omega}$ 구하라.

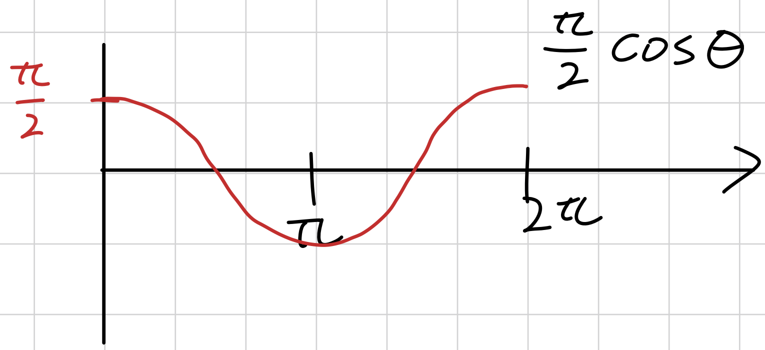
$$\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\sin\theta} \xrightarrow{kd=\pi \text{ 대입}} \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

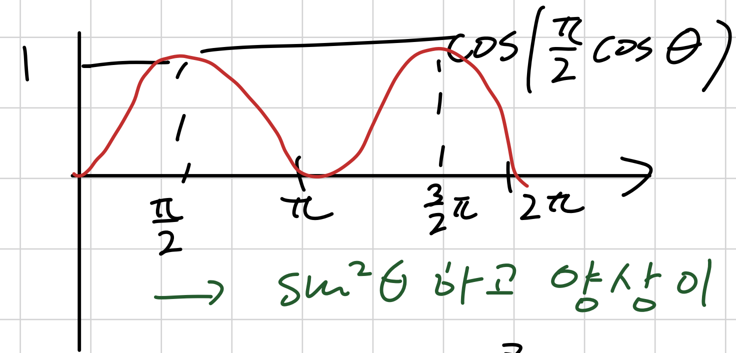
$$\xrightarrow{kd=2\pi \text{ 대입}} \frac{\cos(\pi\cos\theta)+1}{\sin\theta} = \frac{2\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}$$

$$\cos(\pi\cos\theta)+1$$
$$= \cos^2\left(\frac{\pi}{2}\cos\theta\right) - \sin^2\left(\frac{\pi}{2}\cos\theta\right) + 1$$
$$= 2\cos^2\left(\frac{\pi}{2}\cos\theta\right)$$

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\begin{cases} \dfrac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & kd = \pi \\[2ex] \dfrac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} & kd = 2\pi \end{cases}$$

f) 앞선 문제의 angular distribution을 그려라.

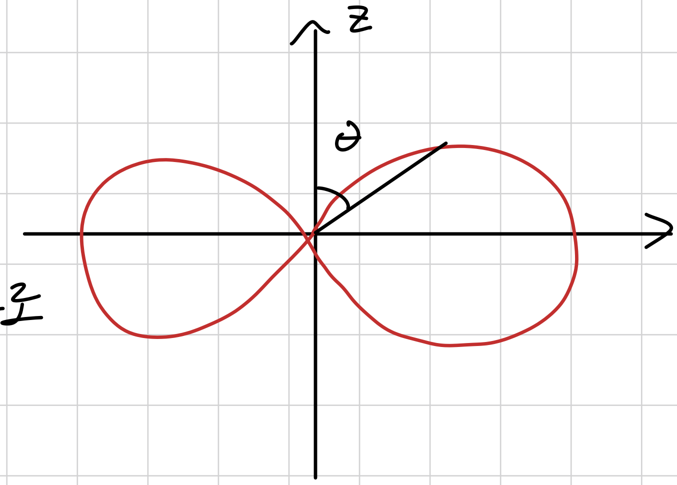




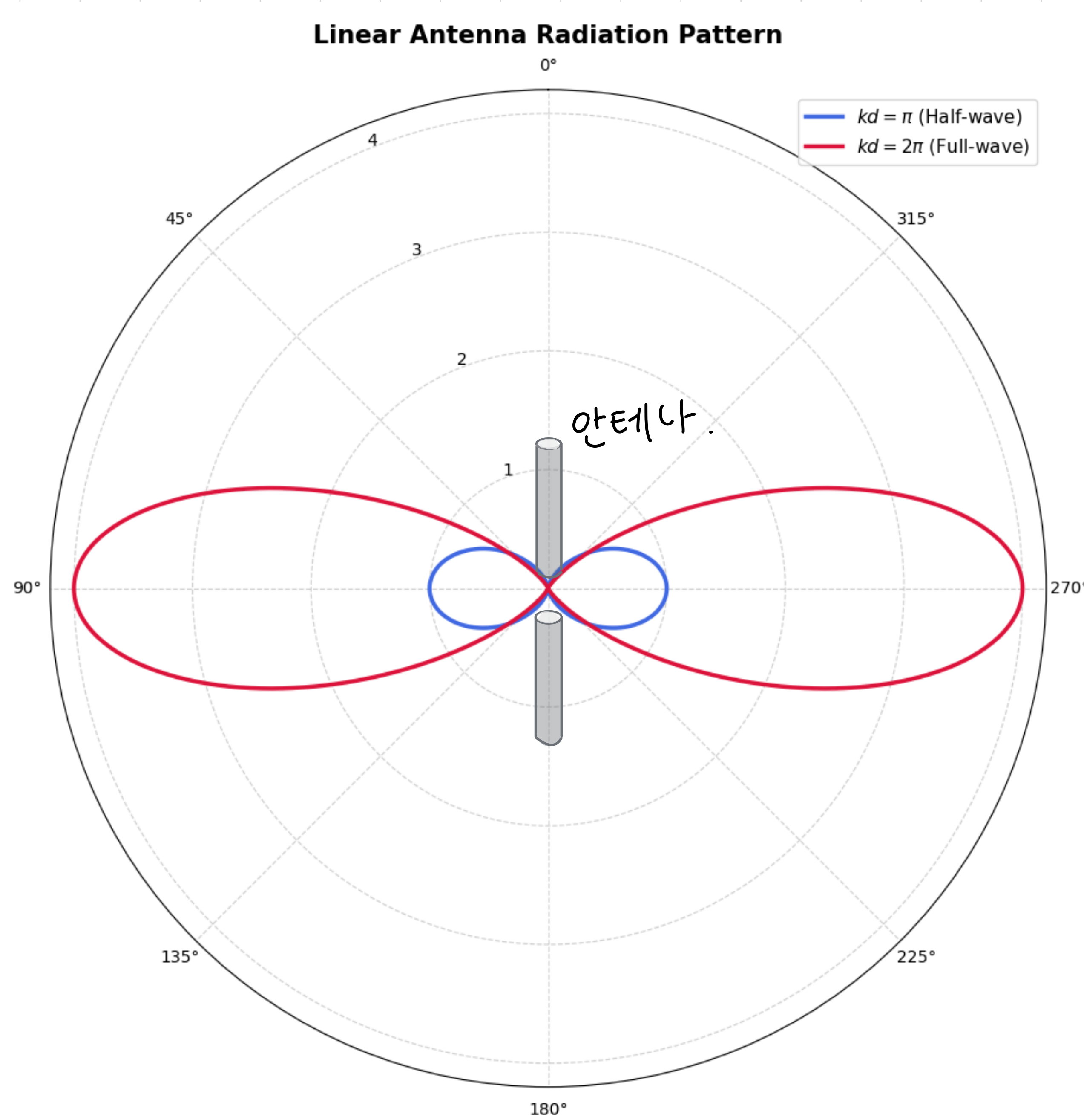
→ $\sin^2\theta$ 하고 양상이 비슷.

$$\frac{dP}{d\Omega} = \frac{Z_0 I^2}{8\pi^2}\begin{cases} \dfrac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \\ \dfrac{4\cos^4\left(\frac{\pi}{2}\cos\theta\right)}{\sin^2\theta} \end{cases}$$

→ 둘 다 이런 모양으로 그려진다.



matplotlib으로 그려서 확인



2. Angular distribution of multipole radiation

$$\frac{dP(l,m)}{d\Omega} = \frac{Z_0}{2k^2} |a(l,m)|^2 |\mathbf{X}_{lm}|^2$$

$$\mathbf{X}_{lm}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}} \mathbb{L} Y_{lm}(\theta,\phi)$$

Y_{lm} 은 spherical harmonics

$$\mathbb{L} = -i(\mathbf{r} \times \nabla)$$

$\ell = 1$ [edit]

$$Y_1^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \quad =$$

$$Y_1^{0}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos\theta \quad =$$

$$Y_1^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \quad =$$

$\ell = 2$ [edit]

$$Y_2^{-2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2\theta$$

$$Y_2^{-1}(\theta,\varphi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin\theta \cdot \cos\theta$$

$$Y_2^{0}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3\cos^2\theta - 1)$$

$$Y_2^{1}(\theta,\varphi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin\theta \cdot \cos\theta$$

$$Y_2^{2}(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2\theta$$

$$Y_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta$$

$$Y_{1\pm1} = \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{\pm i\phi} \sin\theta$$

$$Y_{20} = \frac{1}{4}\sqrt{\frac{5}{\pi}} (3\cos^2\theta - 1)$$

$$Y_{2\pm1} = \mp \frac{1}{2}\sqrt{\frac{15}{2\pi}} e^{\pm i\phi} \sin\theta\cos\theta$$

$$Y_{2\pm2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}} e^{\pm 2i\phi} \sin^2\theta$$

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$$

a) $l=1$, $l=2$ 에 대해 $|\mathbf{X}_{lm}|^2$ 을 구하라.

$$\nabla Y_{10} = -\hat{\theta}\,\frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \frac{1}{r}\sin\theta$$

$$\nabla Y_{1\pm1} = \mp\hat{\theta}\,\frac{1}{2}\sqrt{\frac{3}{2\pi}}\frac{1}{r} e^{\pm i\phi}\cos\theta - \hat{\phi}\,\frac{i}{r\sin\theta}\cdot\frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{\pm i\phi}\sin\theta$$

$$\nabla Y_{20} = -\hat{\theta}\,\frac{3}{2}\sqrt{\frac{5}{\pi}}\frac{1}{r}\cos\theta\sin\theta$$

$$\nabla Y_{2\pm1} = \mp\hat{\theta}\,\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{1}{r} e^{\pm i\phi}\cos 2\theta - \hat{\phi}\,\frac{i}{2}\sqrt{\frac{15}{2\pi}}\frac{1}{r} e^{\pm i\phi}\cos\theta$$

$$\nabla Y_{2\pm2} = \hat{\theta}\,\frac{1}{2}\sqrt{\frac{15}{2\pi}}\frac{1}{r} e^{\pm 2i\phi}\sin\theta\cos\theta \pm \hat{\phi}\,\frac{i}{2}\sqrt{\frac{15}{2\pi}}\frac{1}{r} e^{\pm 2i\phi}\sin\theta$$

$$-i(\mathbf{r}\times\nabla)Y_{10} = i\hat{\phi}\,\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin\theta$$

$$-i(\mathbf{r}\times\nabla)Y_{1\pm1} = \pm i\hat{\phi}\,\frac{1}{2}\sqrt{\frac{3}{2\pi}}\,e^{\pm i\phi}\cos\theta + \hat{\theta}\,\frac{1}{2}\sqrt{\frac{3}{2\pi}}\,e^{\pm i\phi}$$

$$-i(\mathbf{r}\times\nabla)Y_{20} = i\hat{\phi}\,\frac{3}{2}\sqrt{\frac{5}{\pi}}\cos\theta\sin\theta$$

$$-i(\mathbf{r}\times\nabla)Y_{2\pm1} = \pm i\hat{\phi}\,\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,e^{\pm i\phi}\cos 2\theta + \hat{\theta}\,\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,e^{\pm i\phi}\cos\theta$$

$$-i(\mathbf{r}\times\nabla)Y_{2\pm2} = -\hat{\phi}\,\frac{i}{2}\sqrt{\frac{15}{2\pi}}\,e^{\pm 2i\phi}\sin\theta\cos\theta \mp \hat{\theta}\,\frac{1}{2}\sqrt{\frac{15}{2\pi}}\,e^{\pm 2i\phi}\sin\theta$$

$$X_{10} = i\hat{\phi}\,\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta$$

$$X_{1\pm1} = \pm i\hat{\phi}\,\frac{1}{4}\sqrt{\frac{3}{\pi}}\,e^{\pm i\phi}\cos\theta + \hat{\theta}\,\frac{1}{4}\sqrt{\frac{3}{\pi}}\,e^{\pm i\phi}$$

$$X_{20} = i\hat{\phi}\,\frac{3}{2}\sqrt{\frac{5}{6\pi}}\cos\theta\sin\theta$$

$$X_{2\pm1} = \pm i\hat{\phi}\,\frac{1}{4}\sqrt{\frac{5}{\pi}}\,e^{\pm i\phi}\cos 2\theta + \hat{\theta}\,\frac{1}{4}\sqrt{\frac{5}{\pi}}\,e^{\pm i\phi}\cos\theta$$

$$X_{2\pm2} = -\hat{\phi}\,\frac{i}{4}\sqrt{\frac{5}{\pi}}\,e^{\pm 2i\phi}\sin\theta\cos\theta \mp \hat{\theta}\,\frac{1}{4}\sqrt{\frac{5}{\pi}}\,e^{\pm 2i\phi}\sin\theta$$

$$|X_{10}|^2 = \frac{3}{8\pi}\sin^2\theta$$

$$|X_{1\pm1}|^2 = \frac{3}{16\pi}(1+\cos^2\theta)$$

$$|X_{20}|^2 = \frac{15}{8}\cos^2\theta\sin^2\theta$$

$$\cos^2 2\theta = (\cos^2\theta - \sin^2\theta)^2 = (2\cos^2\theta - 1)^2$$

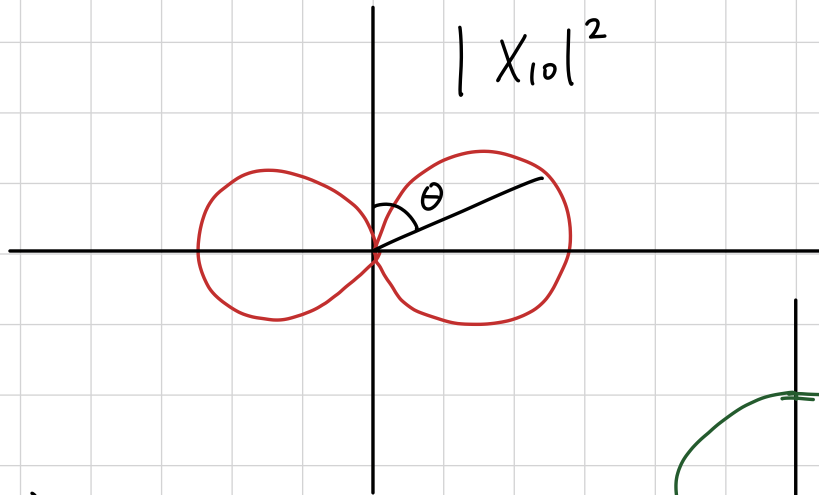
$$|X_{2\pm1}|^2 = \frac{5}{16\pi}\left(\cos^2\theta + \cos^2 2\theta\right) = \frac{5}{16\pi}\left(1 - 3\cos^2\theta + 4\cos^4\theta\right)$$

$$|X_{2\pm2}|^2 = \frac{25}{16\pi}\sin^2\theta\,(1+\cos^2\theta) = \frac{25}{16\pi}(1-\cos^2\theta)(1+\cos^2\theta) = \frac{25}{16\pi}(1-\cos^4\theta)$$

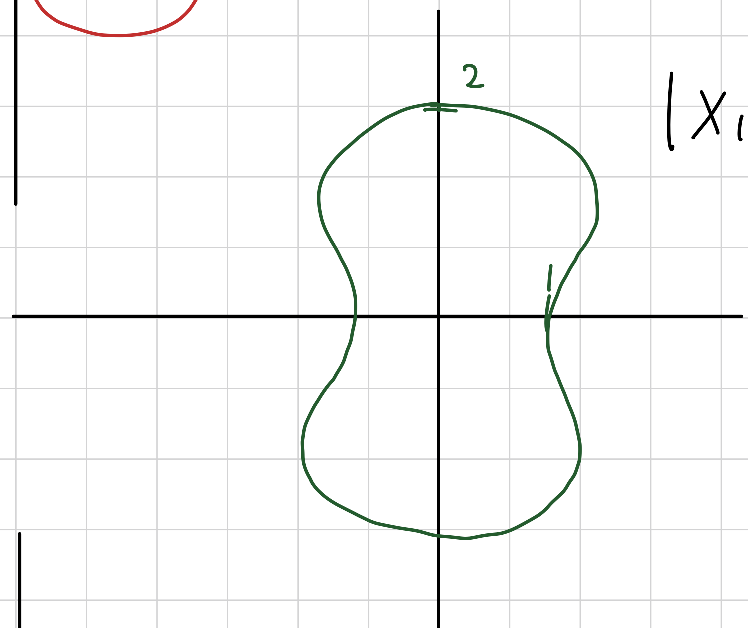
b) Radiation pattern 을 그려라. 먼저 손으로 그려보자.

$|X_{10}|^2 = \frac{3}{8\pi} \sin^2\theta$

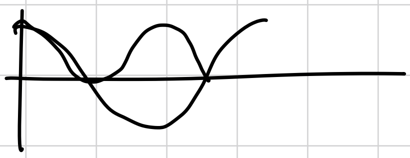
$|X_{10}|^2$



$|X_{1\pm1}|^2 = \frac{3}{16\pi}(1+\cos^2\theta)$

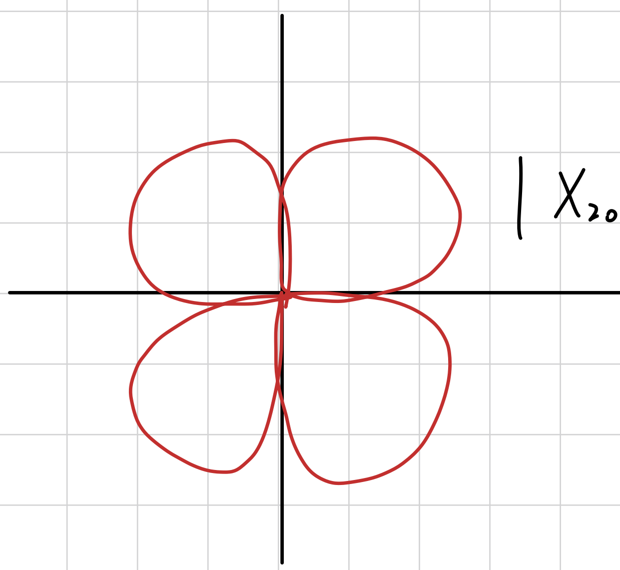


$|X_{1\pm1}|^2$



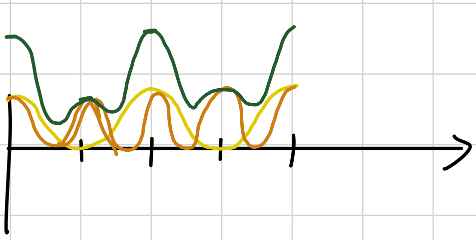
$|X_{20}|^2 = \frac{15}{8} \cos^2\theta \sin^2\theta$

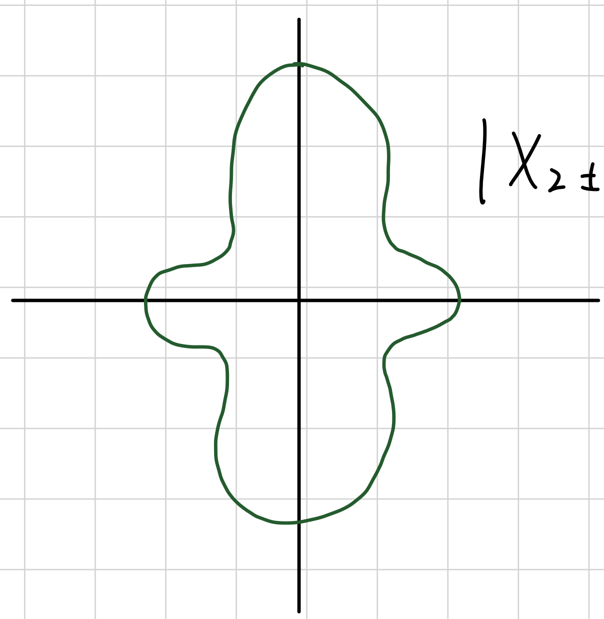
$\frac{1}{2}\sin 2\theta$



$|X_{20}|^2$

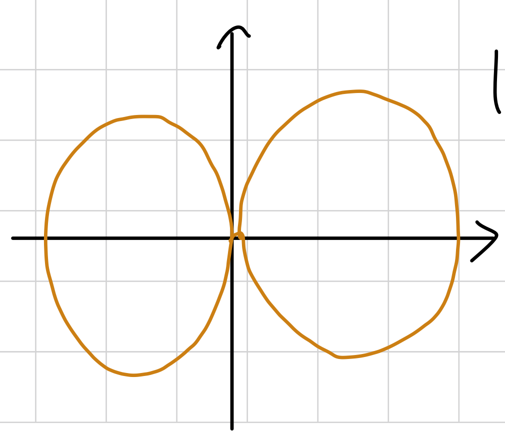
$|X_{2\pm1}|^2 = \frac{5}{16\pi}(\cos^2\theta + \cos^2 2\theta)$





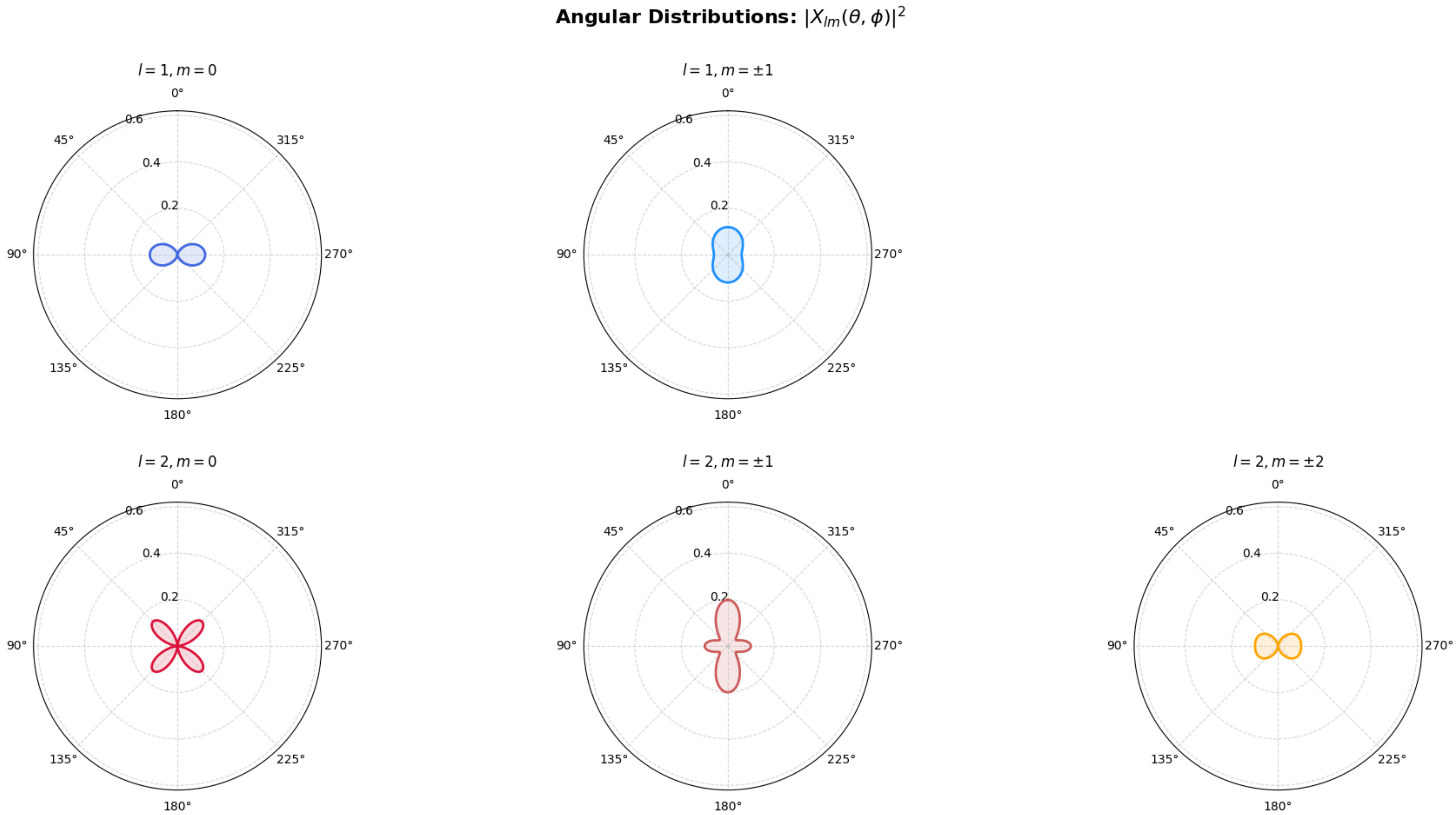
$|X_{2\pm1}|^2$

$|X_{2\pm2}|^2 = \frac{25}{16\pi}(1-\cos^4\theta)$



$|X_{2\pm2}|^2$

Matplotlib 으로 그려봤다.



9.1 A common textbook example of a radiating system (see Problem 9.2) is a configuration of charges fixed relative to each other but in rotation. The charge density is obviously a function of time, but it is not in the form of (9.1).

(a) Show that for rotating charges one alternative is to calculate *real* time-dependent multipole moments using $\rho(\mathbf{x}, t)$ directly and then compute the multipole moments for a given harmonic frequency with the convention of (9.1) by inspection or Fourier decomposition of the time-dependent moments. Note that care must be taken when calculating $q_{lm}(t)$ to form linear combinations that are real before making the connection.

(b) Consider a charge density $\rho(\mathbf{x}, t)$ that is periodic in time with period $T = 2\pi/\omega_0$. By making a Fourier *series* expansion, show that it can be written as

$$\rho(\mathbf{x}, t) = \rho_0(\mathbf{x}) + \sum_{n=1}^{\infty} \text{Re}[2\rho_n(\mathbf{x})e^{-in\omega_0 t}]$$

where

$$\rho_n(\mathbf{x}) = \frac{1}{T}\int_0^T \rho(\mathbf{x}, t)e^{in\omega_0 t}\, dt$$

This shows explicitly how to establish connection with (9.1).

(c) For a single charge q rotating about the origin in the x-y plane in a circle of radius R at constant angular speed ω_0, calculate the $l = 0$ and $l = 1$ multipole moments by the methods of parts a and b and compare. In method b express the charge density $\rho_n(\mathbf{x})$ in cylindrical coordinates. Are there higher multipoles, for example, quadrupole? At what frequencies?

3. Rotating Charges

참고 식) 강의 에서는 안 배웠지만...

Spherical multipole moment

$$Q_{lm} = \int r^l Y_{lm}^* \rho\, d^3x \qquad \text{(Jackson 9.170)}$$

대입

$$\rho(\mathbf{x}, t) = \rho(\mathbf{x}) e^{-i\omega_0 t} \qquad \text{(Jackson 9.1)}$$

$$Q_{lm} = e^{-i\omega_0 t}\int r^l Y_{lm}^* \rho(\mathbf{x})\, d^3x \qquad (\text{식 I})$$

(a) 풀이 →

(Jackson 9.1) 에서 $\rho(\mathbf{x}, t) = \rho(\mathbf{x}) e^{-i\omega t}$ 라고 전하 밀도를 표현한 것과,
문제에서처럼 전하가 실제로 회전하여서 실수값 $\rho(\mathbf{x}, t)$ 가 시간에 대해 주기적으로 달라지는 경우는 다르다. (식 9.1) 은 $\rho(\mathbf{x})$ 로 고정된 분포의 밀도가 $e^{-i\omega t}$ 로 한꺼번에 주기적으로 깜빡이는 모양이기 때문.

문제의 상황처럼 항상 실수값인 $\rho(\mathbf{x}, t)$ 에 spherical multipole moment 의 정의를 적용한다면,

$$q_{lm}(t) = \int r^l Y_{lm}^*(\theta, \phi) \rho(\mathbf{x}, t)\, d^3x$$

(식 I) 꼴로 만들기 위해서 푸리에 변환을 통해 $e^{-i\omega_0 t}$ 를 뽑아야 한다.

전하 분포가 z 축에 대해 ω_0 로 회전한다면, $t=0$ 일 때 ρ_0 와 임의 시간 t 에서 $\rho(t)$의 관계는

$$\rho(r, \theta, \phi, t) = \rho_0(r, \theta, \phi - \omega_0 t)$$

$$q_{lm}(t) = \int r^l Y_{lm}^*(\theta, \phi) \rho_0(r, \theta, \phi - \omega_0 t)\, d^3x$$

$$Y_{lm}^*(\phi, \theta) = e^{-im\phi}\, y_{lm}^*(\theta) \text{ 라고 변수분리}$$

$$q_{lm}(t) = \int r^l e^{-im\phi}\, y_{lm}^*(\theta)\, \rho_0(r, \theta, \phi - \omega_0 t)\, d^3x$$

$\phi' = \phi - \omega_0 t$

$$q_{lm}(t) = e^{-i\omega_0 t}\int r^l e^{-im\phi'}\, y_{lm}^*(\theta)\, \rho_0(r, \theta, \phi')\, d^3x \qquad (\text{식 II})$$

$$= e^{-i\omega_0 t}\int r^l Y_{lm}^*(\theta, \phi')\, \rho_0(r, \theta, \phi')\, d^3x$$

$\rho_0(r,\theta,\phi')$ 는 더이상 t 에 의존하지 않는 분포다.

$e^{-i\omega t}$ 가 추출되면서, (식 II) 는 (식 I) 과 형태가 같아졌고,

$q_{lm}(t)$ 또한 $e^{-i\omega_0 t}$ 로 진동하게 되었다.

그러나 지금 구한 $q_{lm}(t)$ 복소수이다. 그러나 물리적인 time-dependent moment는 실수여야 한다.

이때, $Y_{l,-m} = (-1)^m Y_{lm}^*$ 을 이용하면 $q_{l\ \pm m}$ 을 짝지었을 때 실수 moment 가

되는 것을 알수 있다.

$$q_{l-m}(t) = \int r^l Y_{l-m}^*(\theta,\phi)\,\rho(x,t)\,d^3x$$

$$= (-1)^m \left[\int r^l Y_{lm}^*(\theta,\phi)\,\rho(x,t)\,d^3x\right]^*$$

$$= (-1)^m q_{lm}^*(t)$$

$$q_{lm}(t) + (-1)^m q_{l-m}(t) = q_{lm}(t) + (-1)^{2m} q_{lm}^*(t) = 2\,\mathrm{Re}\left[q_{lm}(t)\right]$$

$$\therefore \frac{1}{2}\left(q_{lm}(t) + (-1)^m q_{l-m}(t)\right) = \cos(\omega_0 t)\int r^l Y_{lm}^*(\theta,\phi')\rho_0(r,\theta,\phi')\,d^3x$$

b) 풀이 → $\rho(x,t) = \rho_0(x) + \sum_{n=1}^{\infty} \mathrm{Re}\left[2\rho_n(x)e^{-i\omega_0 n t}\right]$ 증명

$\rho(x,t)$ 를 푸리에 변환.

$$\rho(x,t) = \sum_{n=-\infty}^{\infty} \rho_n(x)\,e^{-i\omega_0 n t}$$

$e^{-i\omega_0 n t}$ 의 직교성을 이용하면,

$$\frac{1}{T}\int_0^T dt\, e^{-i\omega_0 n t}\, e^{-i\omega_0 m t} = \delta_{nm}$$

이를 이용하면 푸리에 변환식은

$$\rho_n(x) = \frac{1}{T}\int_0^T \rho(x,t)\,e^{in\omega_0 t}\,dt$$

문제에서 주어진 식과 같다.

$\rho(x,t)$ 의 실수성 $\rho^*(x,t) = \rho(x,t)$ 을 이용하면,

$$\rho(x,t) = \rho^*(x,t) = \sum_{n=-\infty}^{\infty} \rho_n^*(x) e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} \rho_n(x) e^{-i\omega_0 n t} = \sum_{n=-\infty}^{\infty} \rho_{-n}(x) e^{i\omega_0 n t}$$

$\therefore \rho_n^*(x) = \rho_{-n}(x)$ Fourier coefficient 에 조건이 생겼다.

$\sum_{n=-\infty}^{\infty}$ 을 n 의 부호를 기준으로 나누자.

$$\rho(x,t) = \rho_0(x,t) + \sum_{n=1}^{\infty} \rho_n(x) e^{-i\omega_0 n t} + \sum_{n=-1}^{-\infty} \rho_n(x) e^{-i\omega_0 n t}$$

$$= \rho_0(x,t) + \sum_{n=1}^{\infty} \left\{ \rho_n(x) e^{-i\omega_0 n t} + \rho_{-n}(x) e^{i\omega_0 n t} \right\}$$

$$= \rho_0(x,t) + \sum_{n=1}^{\infty} \left\{ \rho_n(x) e^{-i\omega_0 n t} + \rho_n^*(x) e^{i\omega_0 n t} \right\}$$

$$= \rho_0(x,t) + \sum_{n=1}^{\infty} \left\{ \rho_n(x) e^{-i\omega_0 n t} + \left(\rho_n(x) e^{-i\omega_0 n t}\right)^* \right\}$$

$$= \rho_0(x,t) + 2\sum_{n=1}^{\infty} \mathrm{Re}\left(\rho_n(x) e^{-i\omega_0 n t}\right)$$

C) 풀이 → (a) 방식을 이용한 풀이

xy 평면, 반지름 R, 각진동수 ω_0 인 전하 밀도식을 spherical coordinate로 나타내면,

$$\rho(x,t) = \frac{q}{R^2} \delta(\cos\theta)\, \delta(\phi - \omega_0 t)\, \delta(r - R)$$

↳ 적분해서 q가 되도록 맞춘 factor

$$\frac{d\cos\theta}{d\theta} = -\sin\theta, \qquad r^2 \sin\theta\, dr\, d\theta\, d\phi = r^2\, dr\, (d\cos\theta)\, d\phi$$

$$q_{lm}(t) = \int r^l Y_{lm}^*(\theta,\phi) \frac{q}{R^2} \delta(\cos\theta)\, \delta(\phi - \omega_0 t)\, \delta(r-R)\, r^2 dr\, (d\cos\theta)\, d\phi$$

$\theta = \frac{\pi}{2}$, $\phi = \omega_0 t$, $r = R$ 대입

$$q_{lm}(t) = R^l Y_{lm}^*\left(\frac{\pi}{2}, \omega_0 t\right) \frac{q}{R^2} R^2 = q R^l Y_{lm}^*\left(\frac{\pi}{2}, \omega_0 t\right)$$

$$\boxed{\begin{aligned} Y_{00} &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\ Y_{10} &= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta \\ Y_{1\pm1} &= \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{\pm i\phi} \sin\theta \end{aligned}}$$

* Monopole ($l=0$, $m=0$)

$$q_{00}(t) = q R^0 Y_{00}^* = \frac{q}{2\sqrt{\pi}}$$

* Dipole ($l=1$, $m=0, \pm1$)

$m=0$ 일 경우, $Y_{10}\left(\frac{\pi}{2}, \omega_0 t\right) = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\frac{\pi}{2} = 0$.

$m=\pm1$, $Y_{1\pm1}\left(\frac{\pi}{2}, \omega_0 t\right) = \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}} e^{\pm i\omega_0 t} \longrightarrow q_{11}(t) = \mp \frac{1}{2}\sqrt{\frac{3}{2\pi}} q R e^{\pm i\omega_0 t}$

(b) 방식풀이

cylindrical coordinate 에서 charge density는

$$\rho(x,t) = \frac{q}{R}\delta(\rho-R)\,\delta(z)\,\delta(\phi-\omega_0 t)$$

푸리에 변환을 해 준다.

$$\rho_n(x) = \frac{1}{T}\int_0^T \frac{q}{R}\delta(\rho-R)\,\delta(z)\,\delta(\phi-\omega_0 t)\,e^{in\omega_0 t}\,dt \quad \rightarrow \frac{1}{\omega_0}d\omega_0 t$$

$\omega_0 t = \phi$ 대입

$$\rho_n(x) = \frac{q}{\omega_0 T R}e^{in\phi}\,\delta(\rho-R)\,\delta(z) = \frac{q}{2\pi R}e^{in\phi}\,\delta(\rho-R)\,\delta(z)$$

앞서 $-n$과 n을 짝지었기 때문에

진동수 $n\omega_0$ harmonics의 effective charge density는 $2\rho_n(x)$ (단, ρ_0는 그대로)

이것을 (식 9.1)의 $\rho^{(n)}(x)$ 처럼 보고 moment를 계산

$d^3x = \rho\,d\rho\,d\phi\,dz$

$$q_{lm}^{(n)} = \int Y_{lm}^*(\theta,\phi)\,r^l\,(2\rho_n(x))\,d^3x$$

$$= \int_0^\infty\int_0^{2\pi}\int_{-\infty}^\infty Y_{lm}^*(\theta,\phi)\,r^l\cdot\frac{q}{\pi R}\,\delta(\rho-R)\,\delta(z)\,e^{in\phi}\rho\,dz\,d\phi\,d\rho$$

dz, $d\rho$ 적분, $z=0$, $\rho=R$, $\theta=\frac{\pi}{2}$ 대입

$$= R^l\frac{q}{\pi R}R\int_0^{2\pi}Y_{lm}^*\left(\frac{\pi}{2},\phi\right)e^{i\phi}\,d\phi$$

$Y_{lm}^*(\phi,\theta) = e^{-im\phi}\,y_{lm}^*(\theta)$ 변수분리

$$= R^l\frac{q}{\pi}y_{lm}^*\left(\frac{\pi}{2}\right)\underline{\int_0^{2\pi}e^{i\phi(n-m)}\,d\phi}$$

orthogonality에 의해

$$= R^l\frac{q}{\pi}y_{lm}^*\left(\frac{\pi}{2}\right)\underline{2\pi\delta_{nm}}$$

$$= 2R^l q\,y_{lm}^*\left(\theta=\frac{\pi}{2}\right)\delta_{nm}$$

즉, $\rho_n(x)$은 $m=n$인 moment에만 기여한다

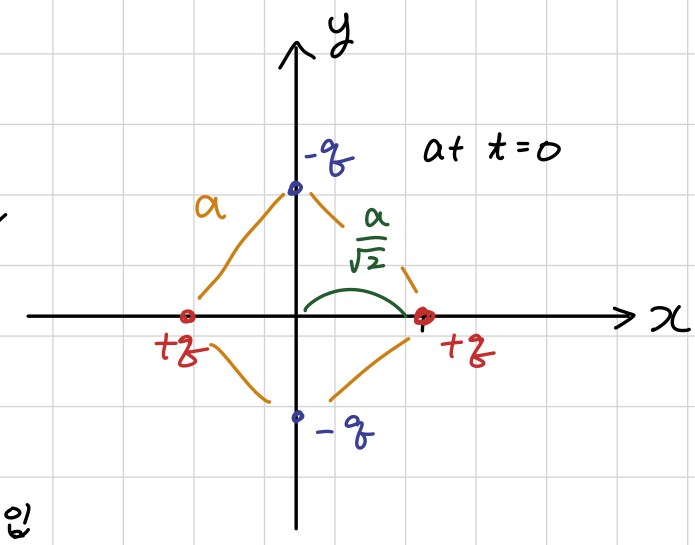
$n=0$은 $l=0, m=0$에 기여. $q_{00}^{(n)} = \int Y_{00}^*(\theta,\phi)\,r^0\rho_0(x)\,d^3x = R^0 q\,y_{00}^*\left(\theta=\frac{\pi}{2}\right) = \frac{q}{2\sqrt{\pi}}$

$n=1$이면 $l=1, m=1$에 기여. $q_{11}^{(n)} = 2Rq\,y_{11}^*\left(\theta=\frac{\pi}{2}\right) = -Rq\sqrt{\frac{3}{2\pi}}$

$l=1$, $m=0$인 경우는 0

9.2 A radiating quadrupole consists of a square of side a with charges $\pm q$ at alternate corners. The square rotates with angular velocity ω about an axis normal to the plane of the square and through its center. Calculate the quadrupole moments, the radiation fields, the angular distribution of radiation, and the total radiated power, all in the long-wavelength approximation. What is the frequency of the radiation?

Quadrupole moment tensor Q_{ij} 를 계산한다.



$t=0$ 일 때 네 전하가 x, y 축에 있는 좌표계를 선택,
전하가 이루는 정사각형 변의 길이가 a. 각 전하는 원점으로부터 $\frac{1}{\sqrt{2}}a$ 떨어진 곳에 있어야 한다.

$+q$ 전하: $\left(\pm\frac{a}{\sqrt{2}}, 0, 0\right)$, $-q$ 전하: $\left(0, \pm\frac{a}{\sqrt{2}}, 0\right)$

Angular frequency ω로 회전, $\begin{pmatrix}\cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t\end{pmatrix}$ 회전 행렬 도입

$+q$ 전하 위치: $\mathbf{x}_+ = \pm\frac{a}{\sqrt{2}}\begin{pmatrix}\cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \pm\frac{a}{\sqrt{2}}\begin{pmatrix}\cos\omega t\\ \sin\omega t\\ 0\end{pmatrix}$

$-q$ 전하 위치: $\mathbf{x}_- = \pm\frac{a}{\sqrt{2}}\begin{pmatrix}\cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \pm\frac{a}{\sqrt{2}}\begin{pmatrix}-\sin\omega t\\ \cos\omega t\\ 0\end{pmatrix}$

Quadrupole moment tensor 의 공식: $\boxed{\underset{k\text{는 }k\text{번째 전하에 대한 index}}{Q_{ij} = \sum_k q_k\left(3x_i^{(k)}x_j^{(k)} - r_{(k)}^2\delta_{ij}\right)}}$

모든 전하에서 $r_{(k)} = \frac{a}{\sqrt{2}}$ 이어서, $\sum_k q_k r_{(k)}^2\delta_{ij} = \frac{a^2}{2}(q+q-q-q)\delta_{ij} = 0$.

$\therefore Q_{ij} = 3\sum_k q_k x_i^{(k)}x_j^{(k)}$ 만 계산하면 된다.

$Q_{xx}(t) = 3\left[2q\left(\frac{a}{\sqrt{2}}\cos\omega t\right)^2 - 2q\left(-\frac{a}{\sqrt{2}}\sin\omega t\right)^2\right] = 3qa^2(\cos^2\omega t - \sin^2\omega t)$

$= 3qa^2\cos 2\omega t$

$Q_{yy}(t) = 3\left[2q\left(\frac{a}{\sqrt{2}}\sin\omega t\right)^2 - 2q\left(\frac{a}{\sqrt{2}}\cos\omega t\right)^2\right] = -3qa^2\cos 2\omega t$

$Q_{xy}(t) = Q_{yx}(t) = 3\left[2q\left(\frac{a}{\sqrt{2}}\cos\omega t\right)\left(\frac{a}{\sqrt{2}}\sin\omega t\right) - 2q\left(-\frac{a}{\sqrt{2}}\sin\omega t\right)\left(\frac{a}{\sqrt{2}}\cos\omega t\right)\right]$

$= 3qa^2\sin 2\omega t$

나머지 z 가 끼인 성분은 0 이다.

모든 Q의 시간에 대한항이 $2\omega t$ 이므로, radiation frequency는 2ω,
wavenumber는 $k=\frac{2\omega}{c}$

$Q_{ij}(t) = Re\left[Q_{ij} e^{-i2\omega t} \right]$ 로 나타내면, 시간 독립적인 성분들은

$$Q_{xx} = 3qa^2 \quad Q_{yy} = -3qa^2 \quad Q_{xy} = Q_{yx} = 3iqa^2$$

이어서 $\longrightarrow$

Radiation field를 계산하자.

Quadrupole 에 의한 magnetic field 는

$$\mathbb{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\,\mathbb{n}\times\mathbb{Q}(\mathbb{n}) \quad (\text{Jackson } 9.44)$$

$\mathbb{Q}(\mathbb{n})$은 벡터, $[\mathbb{Q}(\mathbb{n})]_i = \sum_j Q_{ij} n_j$

구면 좌표계 기준, $\mathbb{n} = \hat{\mathbb{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$[\mathbb{Q}(\mathbb{n})]_x = Q_{xx}\sin\theta\cos\phi + Q_{xy}\sin\theta\sin\phi = 3qa^2(\sin\theta\cos\phi + i\sin\theta\sin\phi)$$
$$= 3qa^2\sin\theta\, e^{i\phi}$$

$$[\mathbb{Q}(\mathbb{n})]_y = Q_{yx}\sin\theta\cos\phi + Q_{yy}\sin\theta\sin\phi = 3qa^2(i\sin\theta\cos\phi - \sin\theta\sin\phi)$$
$$= 3iqa^2\sin\theta\, e^{i\phi}$$

$$[\mathbb{Q}(\mathbb{n})]_z = 0$$

정리하면, $\mathbb{Q}(\mathbb{n}) = 3qa^2\sin\theta\, e^{i\phi}(\hat{x} + i\hat{y})$

spherical coordinate 에서 unit vector로 나타내면,

$$\begin{pmatrix} \hat{x}\cdot\hat{r} & \hat{x}\cdot\hat{\theta} & \hat{x}\cdot\hat{\phi} \\ \hat{y}\cdot\hat{r} & \hat{y}\cdot\hat{\theta} & \hat{y}\cdot\hat{\phi} \\ \hat{z}\cdot\hat{r} & \hat{z}\cdot\hat{\theta} & \hat{y}\cdot\hat{\phi} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

$$\hat{x} = \hat{r}\sin\theta\cos\phi + \hat{\theta}\cos\theta\cos\phi - \hat{\phi}\sin\phi$$
$$\hat{y} = \hat{r}\sin\theta\sin\phi + \hat{\theta}\cos\theta\sin\phi + \hat{\phi}\cos\phi$$

$$\hat{x} + i\hat{y} = \hat{r}(\sin\theta\cos\phi + i\sin\theta\sin\phi) + \hat{\theta}(\cos\theta\cos\phi + i\cos\theta\sin\phi) + \hat{\phi}(i\cos\phi - \sin\phi)$$
$$= \hat{r}\sin\theta\, e^{i\phi} + \hat{\theta}\cos\theta\, e^{i\phi} + \hat{\phi}\, i e^{i\phi}$$

$$\hat{r}\times(\hat{x} + i\hat{y}) = -ie^{i\phi}\hat{\theta} + \cos\theta\, e^{i\phi}\hat{\phi}$$
$$= -ie^{i\phi}(\hat{\theta} + i\cos\theta\,\hat{\phi})$$

$$\hat{r}\times\{\hat{r}\times(\hat{x}+i\hat{y})\} = -ie^{i\phi}\{\hat{r}\times(\hat{\theta} + i\cos\theta\,\hat{\phi})\} = ie^{i\phi}(i\cos\theta\,\hat{\theta} - \hat{\phi})$$

$$\mathbb{H} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}3qa^2\sin\theta\, e^{i\phi}\underbrace{\{-ie^{i\phi}(\hat{\theta} + i\cos\theta\,\hat{\phi})\}}_{\hat{r}\times(\hat{x}+i\hat{y})}$$

$$\mathbb{H} = -\frac{ck^3}{8\pi}\frac{e^{ikr}}{r}qa^2\sin\theta\, e^{i2\phi}(\hat{\theta} + i\cos\theta\,\hat{\phi})$$

$$\mathbb{E} = -c\mu_0(\mathbb{n}\times\mathbb{H}) = -c\mu_0\left(-\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\right)\left(3qa^2\sin\theta\, e^{i\phi}\right)\hat{r}\times\{\hat{r}\times(\hat{x}+i\hat{y})\}$$

$$\mathbb{H} = -\frac{\mu_0 c^2 k^3 a^2}{8\pi}\frac{e^{ikr}}{r}q\sin\theta\, e^{i2\phi}(i\cos\theta\,\hat{\theta} - \hat{\phi})$$

Angular distribution of Radiation을 구해보자.

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 \left| [\mathbb{n} \times \mathbb{Q}(\mathbb{n})] \times \mathbb{n} \right|^2 \qquad \text{(Jackson 9.45)}$$

$$[\mathbb{n} \times \mathbb{Q}(\mathbb{n})] \times \mathbb{n} = \mathbb{Q}(\mathbb{n}) - \mathbb{n}(\mathbb{n} \cdot \mathbb{Q}(\mathbb{n}))$$

$$\left| [\mathbb{n} \times \mathbb{Q}] \times \mathbb{n} \right|^2 = \mathbb{Q}^* \cdot \mathbb{Q} - |\mathbb{n} \cdot \mathbb{Q}|^2$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^2 \left[\mathbb{Q}^* \cdot \mathbb{Q} - |\mathbb{n} \cdot \mathbb{Q}|^2 \right]$$

$$\mathbb{Q}^* \cdot \mathbb{Q} = |Q_x|^2 + |Q_y|^2 = 18 q^2 a^4 \sin^2\theta$$

$$\mathbb{n} \cdot \mathbb{Q} = n_x Q_x + n_y Q_y = 3 q a^2 \sin^2\theta \, e^{2i\phi}$$

$$|\mathbb{n} \cdot \mathbb{Q}|^2 = 9 q^2 a^4 \sin^4\theta$$

$$\mathbb{Q}^* \cdot \mathbb{Q} - |\mathbb{n} \cdot \mathbb{Q}|^2 = 9 q^2 a^4 \sin^2\theta (2 - \sin^2\theta) = 9 q^2 a^4 \sin^2\theta (1 + \cos^2\theta)$$

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{\underbrace{1152\pi^2}} \underbrace{k^6} \, 9 q^2 a^4 \sin^2\theta (1 + \cos^2\theta) = \frac{Z_0 a^4 q^2 \omega^6}{2\pi^2 c^4} \sin^2\theta (1 + \cos^2\theta)$$

$1152 = 9 \cdot 128 = 9 \cdot 2^7$

$k^6 \rightarrow \dfrac{2^6 \omega^6}{c^6}$

Total Radiated power 를 구해보자.

$$d\Omega = \sin\theta \, d\theta \, d\phi$$

$$P = \int \frac{dP}{d\Omega} d\Omega = \frac{Z_0 a^4 q^2 \omega^6}{2\pi^2 c^4} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta (1+\cos^2\theta) d\theta$$

$$= \frac{Z_0 a^4 q^2 \omega^6}{\pi c^4} \int_0^{\pi} \sin^3\theta (1+\cos^2\theta) d\theta$$

$$\int_0^{\pi} \sin^3\theta (1+\cos^2\theta) d\theta = -\int_1^{-1} \sin^2\theta (1+u^2) du = \int_{-1}^{1} (1-u^2)(1+u^2) du = \int_{-1}^{1} (1-u^4) du$$

$u = \cos\theta$, $du = -\sin\theta d\theta$

$$\int_0^{\pi} \sin^3\theta (1+\cos^2\theta) d\theta = \left[u - \frac{1}{5}u^5\right]_{-1}^{1} = \frac{8}{5}$$

$$P = \frac{Z_0 a^4 q^2 \omega^6}{\pi c^4} \cdot \frac{8}{5} = \frac{8 Z_0 a^4 q^2 \omega^6}{5 \pi c^4}$$

9.15 Two fixed electric dipoles of dipole moment p are located in the x-y plane a distance $2a$ apart, their axes parallel and perpendicular to the plane, but their moments directed oppositely. The dipoles rotate with constant angular speed ω about a z axis located halfway between them. The motion is nonrelativistic ($\omega a/c \ll 1$).

(a) Find the lowest nonvanishing multipole moments.

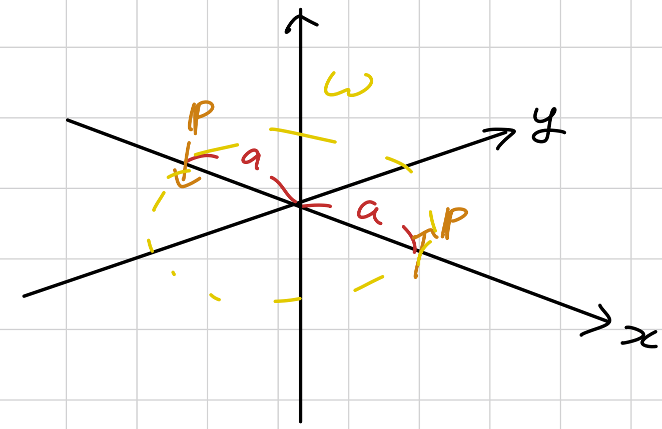
(b) Show that the magnetic field in the radiation zone is, apart from an overall phase factor,

$$\mathbf{H} = \frac{cpa}{2\pi} k^3[(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\cos\theta - \hat{\mathbf{z}}\sin\theta\, e^{i\phi}]\cos\theta\,\frac{e^{ikr}}{r}$$

(c) Show that the angular distribution of the radiation is proportional to $(\cos^2\theta + \cos^4\theta)$ and the total time-averaged power radiated is

$$P = \frac{4}{15\pi\epsilon_0} ck^6 p^2 a^2$$

Hint: Problem 6.21 is relevant.



$t=0$ 에서 두 dipole moment 위치를 이렇게 둔다.

$X_1(t=0) = a\hat{x}$, $X_2(t=0) = -a\hat{x}$

Dipole moment의 방향은

$P_1 = p\hat{z}$ $\quad P_2 = -p\hat{z}$

z 축을 중심으로 회전해도, $P_{1,2}$는 그대로다. X 만 t에 따라 변한다.

$X_1(t) = a(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$

$X_2(t) = -a(\hat{x}\cos\omega t + \hat{y}\sin\omega t)$

system 전체의 frequency는 ω 이며, radiation의 wave number는 $k = \frac{\omega}{c}$ 이다.

a) 풀이 →

Lowest nonvanishing multipole moment를 찾자.

total charge가 0, monopole은 없다. Dipole moment부터 계산하며 찾자.

Total electric dipole moment : $P_{tot} = P_1 + P_2 = p(\hat{z} - \hat{z}) = 0$

대칭으로 있는 두 dipole의 방향이 반대라서 total electric dipole moment도 없다.

Magnetic dipole moment는 있는지 찾아보자.

magnetic dipole moment의 공식은 $\mathbb{m} = \frac{1}{2}\mathbb{P}\times v$

첫번째 dipole에 대해 $\mathbb{m}_1$을 구하면,

$v_1 = \dot{x}_1 = \omega a(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$

$\mathbb{m}_1 = \frac{1}{2}p\omega a\,\hat{z}\times(-\sin\omega t\,\hat{x} + \cos\omega t\,\hat{y})$

$= \frac{1}{2}p\omega a(-\sin\omega t\,\hat{y} - \cos\omega t\,\hat{x})$

$= -\frac{1}{2}p\omega a(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$

두 번째 dipole에 대해서, $v_2 = -v_1$ 이고, $\mathbb{P}_2 = -\mathbb{P}_1$

$\mathbb{m}_2 = \frac{1}{2}\mathbb{P}_2\times v_2 = \frac{1}{2}(-\mathbb{P}_1)\times(-v_1) = \mathbb{m}_1$

Total magnetic dipole moment를 구하면,

$\mathbb{m}(t) = \mathbb{m}_1(t) + \mathbb{m}_2(t) = 2\mathbb{m}_1(t) = -p\omega a(\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y})$

시간의존항을 $e^{-i\omega t}$ 방식으로 나타내자.

$\cos\omega t\,\hat{x} + \sin\omega t\,\hat{y} = \mathrm{Re}\left[(\hat{x} + i\hat{y})e^{-i\omega t}\right]$

$e^{-i\omega t}$를 제외한 $\mathbb{m}$의 complex ampletude는,

$\mathbb{m} = -p\omega a(\hat{x} + i\hat{y}) = -ckap(\hat{x} + i\hat{y})$

... 나중에 $\mathbb{H}_{total}$ 계산이 잘 되려면 여기서 C를 나눠야 한다

$\mathbb{m} = -kap(\hat{x} + i\hat{y})$

Electric quadrupole moment를 구해보자.

위치 $\mathbb{x}_0$의 dipole $\mathbb{P}$의 charge density는 $\rho(\mathbb{x}) = -\mathbb{P}\cdot\nabla\delta(\mathbb{x}-\mathbb{x}_0)$

Quadrupole tensor 를 계산해 보자.

$$Q_{ij} = \int (3x_i x_j - r^2\delta_{ij})\rho(\mathbb{x})\, d^3x = \int (3x_i x_j - r^2\delta_{ij})(-\mathbb{P}\cdot\nabla\delta(\mathbb{x}-\mathbb{x}_0))d^3x$$

디락-델타의 미분을 적분할때 특성.

$$\int_a^b f(x)\left(\frac{d}{dx}\delta(x-x_0)\right)dx = \Big[f(x)\,\delta(x-x_0)\Big]_a^b - \int_a^b\left(\frac{df}{dx}\right)\delta(x-x_0)\,dx = -\left.\frac{df}{dx}\right|_{x=x_0} \quad a < x_0 < b$$

$$Q_{ij} = (-\mathbb{P})\cdot\left\{-\nabla(3x_i x_j - r^2\delta_{ij})\right\}\Big|_{\mathbb{x}=\mathbb{x}_0} = \mathbb{P}\cdot\nabla(3x_i x_j - r^2\delta_{ij})\Big|_{\mathbb{x}=\mathbb{x}_0}$$

$$\left[\nabla(3x_i x_j - r^2\delta_{ij})\right]_l = \partial_l(3x_i x_j - r^2\delta_{ij}) = 3(\delta_{li}x_j + x_i\delta_{lj}) - 2x_l\delta_{ij}$$

$\mathbb{P} = p\hat{z}$ 인 걸 고려,

$$Q_{ij} = p\left[\nabla(3x_i x_j - r^2\delta_{ij})\right]_z\Big|_{\mathbb{x}=\mathbb{x}_0} = p\Big[3(\delta_{zi}x_{0j} + x_{0i}\delta_{zj}) \underbrace{-2x_{0z}\delta_{ij}}\Big]$$

모든 시간에서 $x_{0z}=0$.
이 항은 날아감

$$Q_{ij} = 3p(\delta_{zi}x_{0j} + x_{0i}\delta_{zj})$$

i, j 둘 중 하나는 z여야 $Q_{ij} \neq 0$

첫번째 dipole에 대해, $t=0$ 에서 $x_{01}=a \quad x_{02}=0 \quad x_{03}=0$

$Q^{(1)}_{xz} = Q^{(1)}_{zx} = 3pa$ 나머지 Q는 0

두번째 dipole 에 대해, p와 a 모두 부호를 뒤집으면,

$$Q^{(2)}_{zx} = Q^{(2)}_{xz} = 3(-p)(-a) = 3pa = Q^{(1)}_{xz}$$

time-dependen 한 식을 찾으면,

$$\begin{cases} Q^{(1)}_{xz}(t) = Q^{(1)}_{zx}(t) = Q^{(2)}_{xz}(t) = Q^{(2)}_{zx}(t) = 3pa\cos\omega t = 3pa\,\mathrm{Re}\left[e^{-i\omega t}\right] \\ Q^{(1)}_{yz}(t) = Q^{(1)}_{zy}(t) = Q^{(2)}_{yz}(t) = Q^{(2)}_{zy}(t) = 3pa\sin\omega t = 3pa\,\mathrm{Re}\left[ie^{-i\omega t}\right] \end{cases}$$

$Q^{(1)}$과 $Q^{(2)}$를 더하고, complex ampletude 를 구하면,

$$Q_{xz} = Q_{zx} = 6pa, \qquad Q_{yz} = Q_{zy} = 6ipa$$

magnetic dipole 과 electric quadrupole 모두 k^3 order 의 field 를 만든다.

그래서 둘다 lowest multipole moment 다.

b) 풀이 →

Magnetic Dipolemoment 와 Electric Quadrupole moment 에 의한 $\mathbb{H}$-field는

$$\mathbb{H}_{MD} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}(\mathbb{n}\times\mathbb{m})\times\mathbb{n} = \frac{ck^2}{4\pi}\frac{e^{ikr}}{r}\left[\mathbb{m}-\mathbb{n}(\mathbb{n}\cdot\mathbb{m})\right]$$

$$\mathbb{H}_{EQ} = -\frac{ick^3}{24\pi}\frac{e^{ikr}}{r}\mathbb{n}\times\mathbb{Q}(\mathbb{n}) \qquad \left[\mathbb{Q}(\mathbb{n})\right]_i = \sum_j Q_{ij}n_j$$

$\mathbb{H}_{MD}$ 부터 계산하자. $\mathbb{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\mathbb{n}\cdot\mathbb{m} = -kap(n_x+in_y) = -kap\sin\theta(\cos\phi+i\sin\phi) = -kap\sin\theta e^{i\phi}$$

$$\mathbb{m}-\mathbb{n}(\mathbb{n}\cdot\mathbb{m}) = -kap(\hat{x}+i\hat{y}) + kap\sin\theta e^{i\phi}\mathbb{n}$$

$$\left[\mathbb{m}-\mathbb{n}(\mathbb{n}\cdot\mathbb{m})\right]_x = -kap + kap\sin\theta e^{i\phi}\sin\theta\cos\phi = -kap(1-\sin^2\theta\cos\phi e^{i\phi})$$

$$\left[\mathbb{m}-\mathbb{n}(\mathbb{n}\cdot\mathbb{m})\right]_y = -ikap + kap\sin\theta e^{i\phi}\sin\theta\sin\phi = -kap(i-\sin^2\theta\sin\phi e^{i\phi})$$

$$\left[\mathbb{m}-\mathbb{n}(\mathbb{n}\cdot\mathbb{m})\right]_z = 0 + kap\sin\theta e^{i\phi}\cos\theta = kap\sin\theta\cos\theta e^{i\phi}$$

EQ 에 관한 부분을 계산

$\mathbb{Q}(\mathbb{n})$ 의 각 성분은

$$Q_x = Q_{xz}n_z = 6pa\cos\theta$$

$$Q_y = Q_{yz}n_z = 6ipa\cos\theta$$

$$Q_z = Q_{zx}n_x + Q_{zy}n_y = 6pa\sin\theta\cos\phi + 6ipa\sin\theta\sin\phi = 6pa\sin\theta e^{i\phi}$$

$\mathbb{n} \times \mathbb{Q}(\mathbb{n})$ 을 계산하면,

$$[\mathbb{n}\times\mathbb{Q}]_x = n_y Q_z - n_z Q_y = \sin\theta\sin\phi \cdot 6pa\sin\theta e^{i\phi} - \cos\theta \cdot 6ipa\cos\theta$$

$$= 6pa(\sin^2\theta\sin\phi e^{i\phi} - i\cos^2\theta)$$

$$[\mathbb{n}\times\mathbb{Q}]_y = n_z Q_x - n_x Q_z = \cos\theta\cdot 6pa\cos\theta - \sin\theta\cos\phi\cdot 6pa\sin\theta e^{i\phi}$$

$$= 6pa(\cos^2\theta - \sin^2\theta\cos\phi e^{i\phi})$$

$$[\mathbb{n}\times\mathbb{Q}]_z = n_x Q_y - n_y Q_x = \sin\theta\cos\phi\cdot 6ipa\cos\theta - \sin\theta\sin\phi\cdot 6pa\cos\theta$$

$$= 6pa\sin\theta\cos\theta(i\cos\phi - \sin\phi) = 6ipa\sin\theta\cos\theta e^{i\phi}$$

이제 $\mathbb{H}_{MD}$와 $\mathbb{H}_{EQ}$를 더한다.

$$[\mathbb{H}_{MD}+\mathbb{H}_{EQ}]_x = \frac{ck^3}{4\pi}\frac{e^{ikr}}{r}\left[-ap(1-\sin^2\theta\cos\phi e^{i\phi}) - \frac{i}{\cancel{6}}\cdot\cancel{6}pa(\sin^2\theta\sin\phi e^{i\phi} - i\cos^2\theta)\right]$$

$$ap(-1+\sin^2\theta\cos\phi e^{i\phi} - i\sin^2\theta\sin\phi e^{i\phi} - \cos^2\theta)$$

$$\sin^2\theta e^{-i\phi}e^{i\phi} = \sin^2\theta$$

$$-ap\cdot 2\cos^2\theta$$

$$= \frac{ck^3}{2\pi}\frac{e^{ikr}}{r}(-ap\cos^2\theta)$$

$$[\mathbb{H}_{MD}+\mathbb{H}_{EQ}]_y = \frac{ck^3}{4\pi}\frac{e^{ikr}}{r}\left[-ap(i - \sin^2\theta\sin\phi e^{i\phi}) - \frac{i}{6}6pa(\cos^2\theta - \sin^2\theta\cos\phi e^{i\phi})\right]$$

$$ap(-i + \sin^2\theta\sin\phi e^{i\phi} + i\sin^2\theta\cos\phi e^{i\phi} - i\cos^2\theta$$

$$\sin^2\theta\, i e^{-i\phi+i\phi} = i\sin^2\theta$$

$$iap(-1+\sin^2\theta - \cos^2\theta) = -2iap\cos^2\theta$$

$$= -\frac{ck^3}{2\pi}\frac{e^{ikr}}{r}(iap\cos^2\theta)$$

$$[\mathbb{H}_{MD}+\mathbb{H}_{EQ}]_z = \frac{ck^3}{4\pi}\frac{e^{ikr}}{r}\left[ap\sin\theta\cos\theta e^{i\phi} - \frac{i}{6}6ipa\sin\theta\cos\theta e^{i\phi}\right]$$

$$ap(\sin\theta\cos\theta e^{i\phi} + \sin\theta\cos\theta e^{i\phi})$$

$$= \frac{ck^3}{2\pi}\frac{e^{ikr}}{r}ap\sin\theta\cos\theta e^{i\phi}$$

$$\boxed{\mathbf{H} = \frac{cpa}{2\pi}k^3\left[(\hat{\mathbf{x}}+i\hat{\mathbf{y}})\cos\theta - \hat{\mathbf{z}}\sin\theta\, e^{i\phi}\right]\cos\theta\,\frac{e^{ikr}}{r}}$$

모두 모아 정리하면,

$$\mathbb{H} = \frac{cpa}{2\pi} k^3 \Big[\underbrace{(\hat{x} + i\hat{y})\cos\theta - \hat{z}\sin\theta\, e^{i\phi}}_{\mathbb{V}} \Big] \cos\theta \, \frac{e^{ikr}}{r}$$

C) 풀이 →

$$\frac{dP}{d\Omega} = \frac{1}{2} Z_0 r^2 |\mathbb{H}|^2 \qquad Z_0 = \frac{1}{\varepsilon_0 c}$$

$\mathbb{H}$의 벡터 부분만 보면,

$$\mathbb{V} \equiv \Big[(\hat{x} + i\hat{y})\cos\theta - \hat{z}\sin\theta\, e^{i\phi} \Big] \cos\theta$$

$$|\mathbb{V}|^2 = \cos^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta$$
$$= 2\cos^4\theta + \sin^2\theta\cos^2\theta = \cos^2\theta(\cos^2\theta + 1)$$

$$\frac{dP}{d\Omega} = \frac{cp^2a^2k^2}{8\pi^2\varepsilon_0} \cos^2\theta (\cos^2\theta + 1)$$

Total radiation power 를 구하면,

$$P = \int_0^{2\pi} d\phi \int_0^{\pi} \frac{dP}{d\Omega} \sin\theta\, d\theta = 2\pi \cdot \frac{cp^2a^2k^6}{8\pi^2\varepsilon_0} \int_0^{\pi} (\cos^2\theta + \cos^4\theta)\sin\theta\, d\theta$$

$u = \cos\theta$

$$\int_0^{\pi} (\cos^2\theta + \cos^4\theta)\sin\theta\, d\theta = \int_{-1}^{1} (u^2 + u^4)\, du = \left[\frac{u^3}{3} + \frac{u^5}{5} \right]_{-1}^{1} = \frac{2}{3} + \frac{2}{5} = \frac{16}{15}$$

$$P = 2\pi \cdot \frac{cp^2a^2k^6}{8\pi^2\varepsilon_0} \cdot \frac{16}{15} = \frac{4}{15\pi\varepsilon_0} ck^6p^2a^2$$