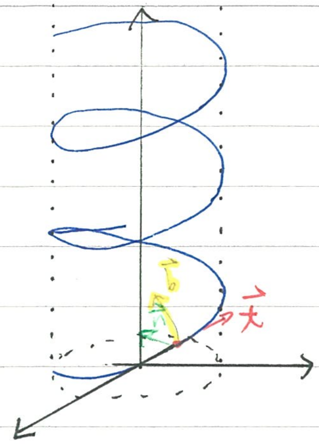
CM Hw 0 1페이지 2번 문제

t : time $\vec{t}$: tangent.



$\frac{d\vec{X}}{dt} = (-a\sin t, a\cos t, c)$

$\frac{ds}{dt} = \left\|\frac{d\vec{x}}{dt}\right\| = \sqrt{a^2+c^2}$

$\frac{d\vec{t}}{dt} = \frac{1}{\sqrt{a^2+c^2}}(-a\cos t, -a\sin t, 0)$

$\vec{X}(t) = (a\cos t, a\sin t, ct)$

tangent $\vec{t}$.

$\vec{t} = \frac{\frac{d\vec{X}}{dt}}{\left\|\frac{d\vec{X}}{dt}\right\|} = \frac{1}{\sqrt{a^2+c^2}}(-a\sin t, a\cos t, c)$

curvature κ

$\kappa = \left\|\frac{d\vec{t}}{dt}\frac{dt}{ds}\right\| = \frac{a}{\sqrt{a^2+c^2}} \times \frac{1}{\sqrt{a^2+c^2}} = \frac{a}{a^2+c^2}$

principal normal $\vec{n}$.

$\vec{n} = \frac{\frac{d\vec{t}}{dt}}{\left\|\frac{d\vec{t}}{dt}\right\|} = \frac{1}{a}(-a\cos t, -a\sin t, 0) = (-\cos t, -\sin t,$

binormal $\vec{b}$.

$$\vec{b} = \vec{t} \times \vec{n} = \frac{1}{\sqrt{a^2+c^2}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin t & a\cos t & c \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{a^2+c^2}}(c\sin t, -c\cos t, a)$$

torsion τ

$\tau = -\left(\frac{d\vec{b}}{dt}\frac{dt}{ds}\right)\cdot\vec{n} = \frac{1}{a^2+c^2}(c\cdot\cos t, c\sin t, 0)\cdot(-\cos t, -\sin$

$= \frac{c}{a^2+c^2}$

left-handed helix 는 어떤가?

$\frac{d\vec{x}}{dt} = (-a\sin t, -a\cos t, c)$

$\vec{X}(t) = (a\cos t, -a\sin t, ct)$

tangent $\vec{t} = \frac{1}{\sqrt{a^2+c^2}}(-a\sin t, -a\cos t, c)$

$\frac{d\vec{t}}{dt} = \frac{1}{\sqrt{a^2+c^2}}(-a\cos t, +a\sin t, 0)$

curvature : $\kappa = \frac{a}{a^2+c^2}$

principal normal $\vec{n} = (-\cos t, \sin t, 0)$

binormal $\vec{b} = \frac{1}{\sqrt{a^2+c^2}}(-c\sin t, -c\cos t, -a)$

$\frac{d\vec{b}}{dt} \propto (-c\cos t, c\sin t, 0)$

torsion $\tau = \left(\frac{1}{a^2+c^2}(-c\cos t, c\sin t, 0)\cdot(-\cos t, \sin t, 0)\right)\times(-1)$

$= \frac{-c}{a^2+c^2}$

torsion는 감길 방향으로 부호가 결정.

CM HW 0   1 페이지 3번 문제

$\mathbb{X}(t) = (x(t), y(t), 0)$, curvature $K(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}$ 를

위 식을 유도하라.

풀이 → $\vec{T}$: tangent vector, s : arc length.

$$\frac{ds}{dt} = \sqrt{\dot{x}^2+\dot{y}^2}, \qquad \vec{T} = \frac{d\mathbb{X}}{ds} = \frac{d\mathbb{X}}{dt}\cdot\frac{dt}{ds} = (\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}\underbrace{(\dot{x}, \dot{y}, 0)}_{\text{vector}}$$

$$K = \left\|\frac{d\vec{T}}{ds}\right\| = \left\|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right\| = (\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}\left\|\frac{d\vec{T}}{dt}\right\|.$$

$$\left\|\frac{d\vec{T}}{dt}\right\| = \left\| \underbrace{(\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}(-\dot{x}\ddot{x}-\dot{y}\ddot{y})}_{\text{scalar part}}\underbrace{(\dot{x}, \dot{y}, 0)}_{\text{vector part}} + \underbrace{(\dot{x}^2+\dot{y}^2)^{-\frac{1}{2}}}_{\text{scalar part}}\underbrace{(\ddot{x}, \ddot{y}, 0)}_{\text{vector part}} \right\|$$

$$= \left\|(\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}\underbrace{(-\dot{y}\ddot{y}\dot{x}+\dot{y}^2\ddot{x},\ -\dot{x}\dot{y}\ddot{x}+\dot{x}^2\ddot{y},\ 0)}_{\text{vector part.}}\right\|$$

$$= (\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}\left[\dot{y}^2(-\dot{x}\ddot{y}+\dot{y}\ddot{x})^2 + \dot{x}^2(-\dot{y}\ddot{x}+\dot{x}\ddot{y})^2\right]^{\frac{1}{2}}$$

$$= (\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}\left[(\dot{x}^2+\dot{y}^2)(-\dot{x}\ddot{y}+\dot{y}\ddot{x})^2\right]^{\frac{1}{2}}$$

$$= (\dot{x}^2+\dot{y}^2)^{-1}\,|\dot{x}\ddot{y}-\dot{y}\ddot{x}|.$$

$K = (\dot{x}^2+\dot{y}^2)^{-\frac{3}{2}}\,|\dot{x}\ddot{y}-\dot{y}\ddot{x}|$ → what I expected.

---

ellipse 의 최대 최소 곡률은 구하라 $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

t로 매개변수표현, $x(t) = a\cos t$, $y(t) = b\sin t$.

$\dot{x} = -a\sin t$, $\ddot{x} = -a\cos t$, $\dot{y} = b\cos t$, $\ddot{y} = -b\sin t$.

$K = \dfrac{|ab|}{a^2\sin^2 t + b^2\cos^2 t}$ → t로 미분해준다. 분자가 $(2b^2-2a^2)\sin t\cos t$ 에 비

$\sin t = 0$ or $\cos t = 0$ 일때 극값이다.

K의 극값은 $(x, y) = (a, 0)$ or $(0, b)$ 일 때 나온다.
$\qquad\qquad\qquad\quad (-a, 0) \qquad (0, -b)$

따라서 $\dfrac{b}{a}$ 와 $\dfrac{a}{b}$ 가 곡률의 최대 아니면 최소 값.

$b > a$ 면 $\dfrac{b}{a}$ 가 최대겠지.

CM HW0 1페이지 4번 문제

(i) $t \in [0,1]$ $\vec{f}(t) = (t, t, t) = (x(t), y(t), z(t))$.

(ii) $t \in [0,1]$ $\vec{g}(t) = (t, t, t^2)$

for (i). $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1$, $dx = dy = dz = dt$

$$\int_{\vec{f}} (x\,dx + y\,dy + z\,dz) = \int_0^1 3t\,dt = \frac{3}{2}$$

$$\int_f (y\,dx + x\,dy + dz) = \int_0^1 (2t+1)\,dt = 2$$

$$\int_f (y\,dx - x\,dy + e^{x+y} dz) = \int_0^1 e^{2t}\,dt = \frac{1}{2}(e^2 - 1)$$

for (ii) $\frac{dx}{dt} = \frac{dy}{dt} = 1$, $\frac{dz}{dt} = 2t$, $\begin{cases} dx = dy = dt \\ dz = 2t\,dt. \end{cases}$

$$\int_g (x\,dx + y\,dy + z\,dz) = \int_0^1 (t + t + t^2 \cdot 2t)\,dt$$
$$= \int_0^1 (2t + 2t^3)\,dt$$
$$= \left[t^2 + \frac{1}{2}t^4\right]_0^1 = \frac{3}{2}$$

$$\int_g (y\,dx + x\,dy + dz) = \int_0^1 (t + t + 2t)\,dt$$
$$= \int_0^1 4t\,dt$$
$$= 2$$

$$\int_g (y\,dx - x\,dy + e^{x+y} dz) = \int_0^1 (t - t + e^{2t} \cdot 2t)\,dt$$
$$= \int_0^1 2te^{2t}\,dt$$
$$= \left[te^{2t}\right]_0^1 - \int_0^1 e^{2t}\,dt$$
$$= e^2 - (e^2 - 1) = 1$$

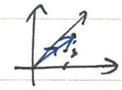
CM HW0

1페이지
5번 문제

$\mathbb{F}(\mathbb{x}) = (3x^2yz^2,\ 2x^3yz,\ x^3z^2)$ $\quad\quad$ $\mathbb{G}(\mathbb{x}) = (3x^2y^2z,\ 2x^3yz,\ x^3y^2)$

세 가지의 직선 경로.

(i) $(0,0,0) \to (1,1,1)$



$C_1 = (t,t,t)$, $\quad t \in [0,1]$

(ii) $(0,0,0) \xrightarrow{C_{2a}} (0,0,1) \xrightarrow{C_{2b}} (0,1,1) \xrightarrow{C_{2c}} (1,1,1)$.

$C_{2a} = (0,0,t)$ $\quad t \in [0,1]$

$C_{2b} = (0,t,1)$ $\quad t \in [0,1]$

$C_{2c} = (t,1,1)$ $\quad t \in [0,1]$

(iii) $(0,0,0) \xrightarrow{C_{3a}} (1,0,0) \xrightarrow{C_{3b}} (1,1,0) \xrightarrow{C_{3c}} (1,1,1)$

$C_{3a} = (t,0,0)$, $C_{3b} = (1,t,0)$, $C_{3c} = (1,1,t)$ $\quad t \in [0,1]$

$\mathbb{F}(\mathbb{x})$ 에 대해 계산.

(i) $$\int_{C_1} \mathbb{F}\cdot d\mathbb{x} = \int_{C_1} (3x^2yz^2dx + 2x^3yzdy + x^3z^2dz)$$

$$= \int_0^1 (3t^5dt + 2t^5dt + t^5dt)$$

$$= 6\int_0^1 t^5dt = 1$$

(ii) $$\int_{C_{2a}} \mathbb{F}\cdot d\mathbb{x} + \int_{C_{2b}} \mathbb{F}\cdot d\mathbb{x} + \int_{C_{2c}} \mathbb{F}\cdot d\mathbb{x}$$

$$= \int_{C_{2a}} x^3z^2dz + \int_{C_{2b}} 2x^3yzdy + \int_{C_{2c}} 3x^2yz^2dx$$

$$= 0 + 0 + \int_0^1 3t^2\cdot 1\cdot 1^2 dt = 1$$

(iii) $$\int_{C_{3a}} \mathbb{F}\cdot d\mathbb{x} + \int_{C_{3b}} \mathbb{F}d\mathbb{x} + \int_{C_{3c}} \mathbb{F}\cdot d\mathbb{x}$$

$$= \int_{C_{3a}} 3x^2yz^2dx + \int_{C_{3b}} 2x^3yzdy + \int_{C_{3c}} x^3z^2dz$$

$$= 0 + 0 + \int_0^1 z^2dt = \frac{1}{3}$$

$\mathbb{G}(\mathbb{x})$ 에 대해 계산

(i) $$\int_{C_1} \mathbb{G}\cdot d\mathbb{x} = \int_{C_1} (3x^2y^2zdx + 2x^3yzdy + x^3y^2dz)$$

$$= \int_0^1 6t^5dt = 1$$

(ii) $$\int_{C_{2a}} \mathbb{G}\cdot d\mathbb{x} + \int_{C_{2b}} \mathbb{G}\cdot d\mathbb{x} + \int_{C_{2c}} \mathbb{G}\cdot d\mathbb{x}$$

$$= 0 + 0 + \int_{C_{2c}} 3x^2y^2zdx$$

$$= \int_0^1 3t^2dt = 1$$

(iii) $$\int_{C_{3a}} \mathbb{G}\cdot d\mathbb{x} + \int_{C_{3b}} \mathbb{G}\cdot d\mathbb{x} + \int_{C_{3c}} \mathbb{G}\cdot d\mathbb{x}$$

$$= 0 + 0 + \int_{C_{3c}} x^3y^2dz$$

$$= \int_0^1 1\,dt = \frac{1}{2}$$

CM HW0 2페이지 6번 문제

A curve C.

$\mathbb{X}(t) = \left(\cos(\sin nt)\cos t,\ \cos(\sin nt)\sin t,\ \sin(\sin nt) \right) \qquad t \in [0, 2\pi]$

$\mathbb{F}(\mathbb{x}) = \left(-\frac{y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ 0 \right)$

$\oint_C \mathbb{F} \cdot d\mathbb{x} = 2\pi$ 임을 보여라. $\mathbb{F}(\mathbb{x})$가 scalar function 의 gradient로 쓰일수 있나.

~~Sho~~ spherical polar coordinate를 써라. $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

폐곡선으로 적분했을 때 값이 0이 아니란 건 $\nabla \times \mathbb{F} \neq 0$ 이란 뜻

-어떤 스칼라 함수의 curl은 무조건 0 이므로, $\mathbb{F}$는 gradient로 쓰일수 없다.

Curve $\mathbb{X}$를 스케치하면,

$\{r(t)\}^2 = x^2(t) + y^2(t) + z^2(t) = \cos^2(\sin nt)\{\cos^2 t + \sin^2 t\} + \sin^2(\sin t)$

$= 1.$

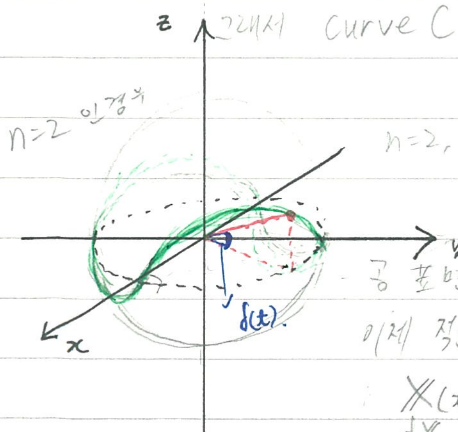
$r(t) = 1$

$\cos\theta(t) = z(t) = \sin(\sin nt) = \sin(-\theta + \frac{\pi}{2})$

$\therefore \theta(t) = \frac{\pi}{2} - \sin nt \qquad$ Let $\delta(t) = -\sin nt$, $\theta(t) = \frac{\pi}{2} + \delta(t)$

$x(t) = \sin(\theta(t))\cos(\phi(t)) = \sin(\frac{\pi}{2} - \sin nt)\cos(\phi(t)) = \cos(\sin nt)\cos t$

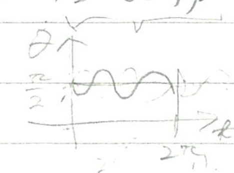
$\therefore \phi(t) = t$



그래서 curve C를 r, ϕ, θ로 나타내면, $\mathbb{X}(t) = (r(t), \phi(t), \theta(t)) = (1, t, \frac{\pi}{2} + \delta(t))$

$\mathbb{X}(0) = (1, 0, 0)$

$n=2,\ \mathbb{X}(\frac{\pi}{2}) = (0, 1, 0)$



-공 표면에 n개 단위의 굴절 ∿ 무늬가 있는 듯한 경로.

이제 적분을 구해보면,

$\mathbb{X}(t) = (\cos\delta\cos t,\ \cos\delta\sin t,\ \sin\delta)$

$\frac{d\mathbb{X}}{dt} = \left(-\cos\delta\sin t - (\sin\delta\cos t)\frac{d\delta}{dt},\ \cos\delta\cos t - (\sin\delta\sin t)\frac{d\delta}{dt},\ (\cos\delta)\frac{d\delta}{dt} \right)$

$\mathbb{F}(t) = \left(-\frac{\sin t}{\cos\delta},\ \frac{\cos t}{\cos\delta},\ 0 \right)$

$\left(\mathbb{F} \cdot \frac{d\mathbb{X}}{dt}\right)dt = \left\{ \sin^2 t + (\tan\delta\sin t\cos t)\frac{d\delta}{dt} + \cos^2 t - (\tan\delta\sin t\cos t)\frac{d\delta}{dt} \right\} dt$

$= dt.$

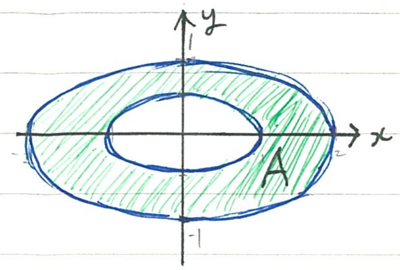
$\oint_C \mathbb{F} \cdot d\mathbb{X} = \int_0^{2\pi} \left(\mathbb{F} \cdot \frac{d\mathbb{X}}{dt}\right) dt = \int_0^{2\pi} dt = 2\pi.$

CM HW0 2페이지 7번문제

$\int_D \frac{x^2}{x^2+4y^2}\, dA$ 를 구하라.

D는 두 타원 $x^2+4y^2=1$, $x^2+4y=4$ 사이 영역.

$x = r\cos\phi$ $y=\frac{1}{2} r\sin\phi$ 의 치환을 이용하라.



$x^2+4y^2=r^2$ A 영역에서, $r\in[1,2]$, $\phi\in[0,2\pi]$

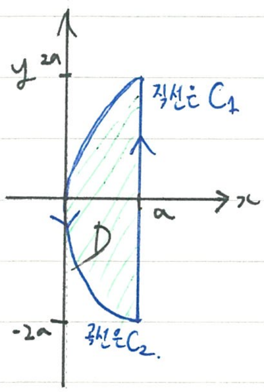
$dA = r\,d\phi\,dr$

$$\int_D \frac{r^2\cos^2\phi}{r^2}\, dA = \int_1^2\int_0^{2\pi} r\cos^2\phi\, d\phi\, dr = \frac{1}{2}(4-1)\times\left(\frac{2\pi}{2}\right) = \frac{3}{2}\pi$$

CM HW0 2페이지 8번 문제

곡선 $C = C_1 + C_2$.

$$\begin{cases} C_1 = (x(t), y(t)) = (a, 2at) & t\in[-1,1] \\ C_2 = (x(t), y(t)) = (at^2, -2at) & t\in[-1,1] \end{cases}$$



$$\begin{cases} C_1\text{의 경우, } \frac{dx}{dt}=0,\ \frac{dy}{dt}=2a \\ C_2\text{의 경우, } \frac{dx}{dt}=2at,\ \frac{dy}{dt}=-2a \end{cases}$$

$$\oint_C (x^2y\,dx + xy^2\,dy) = \int_{C_1}(x^2y\,dx + xy^2\,dy) + \int_{C_2}(x^2y\,dx + xy^2\,dy)$$

$$= \int_{-1}^1 4a^3t^2\cdot 2a\,dt + \int_{-1}^1 -2a^3t^5\cdot 2at\,dt + \int_{-1}^1 4a^3t^4\cdot(-2a)\,dt$$

$$= 8a^4\int_{-1}^1 t^2\,dt - 4a^4\int_{-1}^1 t^6\,dt - 8a^4\int_{-1}^1 t^4\,dt$$

$$= 2\left(8\cdot\frac{1}{3} - 4\cdot\frac{1}{7} - \frac{8}{5}\right)a^4 = 2\cdot\frac{8\cdot7\cdot5 - 4\cdot3\cdot5 - 8\cdot3\cdot7}{105}a^4 = \frac{104}{105}a^4$$

$F = (x^2y, xy^2, 0)$, $\nabla\times F = (0, 0, y^2-x^2)$

$\oint_C F\cdot dx = \int_D \nabla\times F\, dA$,

$$\int_D (y^2-x^2)\,dA = \int_0^a\int_{-2\sqrt{ax}}^{2\sqrt{ax}} (y^2-x^2)\,dy\,dx = \int_0^a\left(\frac{16}{3}a^{\frac{3}{2}}x^{\frac{3}{2}} - 4a^{\frac{1}{2}}x^{\frac{5}{2}}\right)dx$$

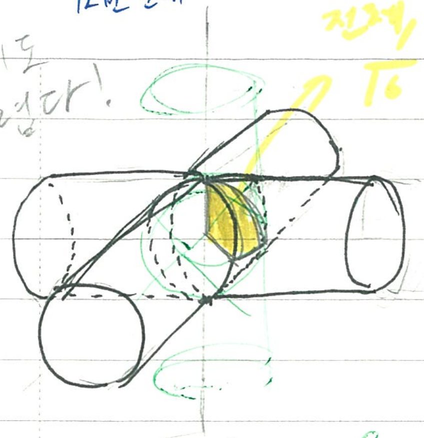
$$= \left[\frac{16}{3}\cdot\frac{2}{5}a^{\frac{3}{2}}x^{\frac{5}{2}} - 4a^{\frac{1}{2}}\cdot\frac{2}{7}a^{\frac{7}{2}}\right]_0^a$$

$$= a^4\left(\frac{16}{3}\times\frac{2}{5} - 4\times\frac{2}{7}\right)$$

$$= \frac{104}{105}a^4$$

CM HW0 2페이지 12번 문제

그리기도 어렵다!

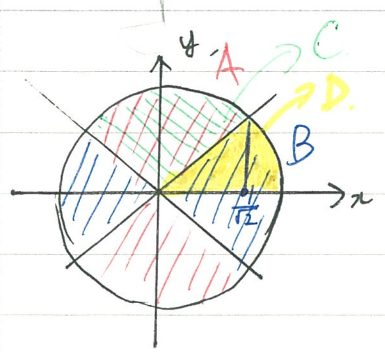


$V = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le 1,\ y^2+z^2 \le 1,\ z^2+x^2 \le 1\}$

이것의 부피를 구하라.

이 도형의 표면에 있는 점에서 $z(x,y)$ 가 무엇인지 파악하자. $x^2+y^2 \le 1$

$z^2(x,y) + y^2 = 1$ or $z^2(x,y) + x^2 = 1$



$x=y$ 와 $x=-y$ 로 경계가 나뉜.

$x^2+y^2 \le 1$ 내부 구역

$A = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1,\ |x| < |y|\}$

$B = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\ \ |y| < |x|\}$

A 에서는 $z^2 = 1-y^2$, B 에서는 $z^2 = 1-x^2$

그럼, C ... 도 아니고! D 영역에서 적분한 다음 8배하면 V 다.

이것의 적분의 부피를 구해

$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0,\ x > y\}$

$$y(x) = \begin{cases} x & (0 < x < \frac{1}{\sqrt{2}}) \\ \sqrt{1-x^2} & (\frac{1}{\sqrt{2}} < x < 1) \end{cases}$$

$z(x) = \sqrt{1-x^2} \quad (0 < x < 1)$

$$\frac{V}{16} = \int_0^{\frac{1}{\sqrt{2}}}\int_0^{x}\int_0^{\sqrt{1-x^2}} 1\, dz\,dy\,dx + \int_{\frac{1}{\sqrt{2}}}^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2}} dz\,dy\,dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x\sqrt{1-x^2}\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} (1-x^2)\,dx$$

$k = x^2$

$$= \int_0^{\frac{1}{2}} \frac{1}{2}\sqrt{1-k}\,dk + \left[x - \frac{1}{3}x^3\right]_{\frac{1}{\sqrt{2}}}^{1} = \frac{1}{3}\left(1 - 2^{-\frac{3}{2}}\right) + \left(1 - \frac{1}{3} - 2^{-\frac{1}{2}} + \frac{1}{3}2^{-\frac{3}{2}}\right)$$

$$= 1 - \frac{1}{\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}}$$

$V = 8\sqrt{2}(\sqrt{2}-1) = 16 - 8\sqrt{2}$

CM HWO 2페이지 2번 문제.

vectorfield $\mathbb{B}(x)$ 는 $\overbrace{}^{\text{모든 곳에서}}$ $f(x) = \text{constant}$ 인 family of surface 의 normal 에 평행하다. $\mathbb{B} \cdot (\nabla \times \mathbb{B}) = 0$ 임을 보여라.

→ 풀이

$\mathbb{B}(x)$ 는 바로 $\nabla f(x)$ 와 평행한 벡터장이다.

~~일반화하면~~, normals of a family of surfaces 가 $\nabla f(x)$ 와 평행하기 때문이다. 평행한 벡터는 어떤 벡터에 스칼라를 곱한 것이므로,

$\mathbb{B}(x) = \lambda(x) \nabla f(x)$. $\lambda(x)$ 는 스칼라 함수

0. gradient 의 curl은 0 이다.

$$\nabla \times \mathbb{B} = \nabla \times (\lambda(x) \nabla f(x)) = \lambda(x) \cancel{\nabla \times (\nabla f(x))} + (\nabla \lambda(x)) \times (\nabla f(x))$$
$$= (\nabla \lambda(x)) \times (\nabla f(x))$$

$$B \cdot (\nabla \times \mathbb{B}) = (\lambda \nabla f) \cdot \{(\nabla \lambda) \times (\nabla f)\} = \lambda \det \begin{vmatrix} \nabla f_x & \nabla f_y & \nabla f_z \\ \nabla \lambda_x & \nabla \lambda_y & \nabla \lambda_z \\ \nabla f_x & \nabla f_y & \nabla f_z \end{vmatrix}$$

적용

$$A \cdot (B \times C) = \det \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$$

$= 0$.

determinant 에서 행들이 lineary dependent 하면 값은 0 이다.

~~ex~~ Vector field $\mathbb{H}(x)$ 는 어떤 곡선의 tangent vector와 평행하다. 그 곡선의 curvature 이 $K = |\mathbb{H} \times (\mathbb{H} \cdot \nabla)\mathbb{H}| / |\mathbb{H}|^3$ 임을 보여라.

곡선이 $\mathbb{C}(t) = (x(t), y(t), z(t))$ 이라 두고,

tangent vector 는 ① $T(t) = \text{unit}\left(\frac{d\mathbb{C}}{dt}\right) = \left\|\frac{dt}{d\mathbb{C}}\right\| \frac{d\mathbb{C}}{dt}$

curvature $K = \left\|\frac{dT}{ds}\right\|$ s는 arc length $\frac{ds}{dt} = \left\|\frac{d\mathbb{C}}{dt}\right\|$

$\lambda(x) > 0$ 이라 가정.

어떤 스칼라함수 $\lambda(x)$ 를 이용, T 와 평행한 $\mathbb{H}(x)$ 는

$\mathbb{H}(x) = \lambda(x) \frac{d\mathbb{C}}{dt}$ 이다, $\left\|\frac{d\mathbb{C}}{dt}\right\| = \frac{ds}{dt} = \frac{1}{|\lambda|}|\mathbb{H}|$, $\frac{d\mathbb{C}}{dt} = \frac{1}{\lambda}\mathbb{H}$

$T(t) = \left\|\frac{dt}{d\mathbb{C}}\right\| \frac{d\mathbb{C}}{dt} = \frac{\lambda}{|\mathbb{H}|} \cdot \frac{\mathbb{H}}{\lambda} = \frac{\mathbb{H}}{|\mathbb{H}|}$, $\frac{d}{ds} = \frac{\lambda}{|\mathbb{H}|} \frac{d}{dt}$

$$\frac{dT}{ds} = \frac{\lambda}{|\mathbb{H}|} \frac{d}{dt} T = \frac{\lambda}{|\mathbb{H}|} \frac{d}{dt} \frac{\mathbb{H}}{|\mathbb{H}|}$$

$$\frac{d|\mathbb{H}|}{dt} = \frac{d}{dt}(\mathbb{H} \cdot \mathbb{H})^{\frac{1}{2}} = \frac{1}{2|\mathbb{H}|}\left(2\mathbb{H} \cdot \frac{d\mathbb{H}}{dt}\right) = \frac{\mathbb{H} \cdot \frac{d}{dt}\mathbb{H}}{|\mathbb{H}|}$$

$$\frac{d}{dt} \frac{\mathbb{H}}{|\mathbb{H}|} = \frac{1}{|\mathbb{H}|} \frac{d\mathbb{H}}{dt} - \frac{\left(\mathbb{H} \cdot \frac{d\mathbb{H}}{dt}\right)\mathbb{H}}{|\mathbb{H}|^3}$$

$$\frac{d}{dt} = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dz}{dt}\frac{\partial}{\partial z} = \frac{1}{\lambda}(\mathbb{H} \cdot \nabla)$$

$$\frac{dT}{ds} = \frac{\lambda}{|\mathbb{H}|}\frac{d}{d\lambda}\frac{\mathbb{H}}{|\mathbb{H}|}$$

$$\frac{d}{dt}\frac{\mathbb{H}}{|\mathbb{H}|} = \frac{1}{|\mathbb{H}|}\cdot\frac{1}{\lambda}(\mathbb{H}\cdot\nabla)\mathbb{H} - \frac{1}{\lambda|\mathbb{H}|^3}[\mathbb{H}\cdot(\mathbb{H}\cdot\nabla)\mathbb{H}]\mathbb{H}$$

$$= \frac{(\mathbb{H}\cdot\nabla)\mathbb{H}}{\lambda|\mathbb{H}|} - \frac{[\mathbb{H}\cdot(\mathbb{H}\cdot\nabla)\mathbb{H}]\mathbb{H}}{\lambda|\mathbb{H}|^3}$$

$$\frac{dT}{ds} = \frac{(\mathbb{H}\cdot\nabla)\mathbb{H}}{|\mathbb{H}|^2} - \frac{[\mathbb{H}\cdot(\mathbb{H}\cdot\nabla)\mathbb{H}]\mathbb{H}}{|\mathbb{H}|^4}$$

Let. $(\mathbb{H}\cdot\nabla)\mathbb{H} = \mathbb{K}$

$$\frac{dT}{ds} = \frac{\mathbb{K}}{|\mathbb{H}|^2} - \frac{[\mathbb{H}\cdot\mathbb{K}]\mathbb{H}}{|\mathbb{H}|^4} = \frac{[\mathbb{H}\cdot\mathbb{H}]\mathbb{K} - [\mathbb{H}\cdot\mathbb{K}]\mathbb{H}}{|\mathbb{H}|^4}$$

$$= \frac{\mathbb{H}\times(\mathbb{K}\times\mathbb{H})}{|\mathbb{H}|^4}$$

어떤 벡터 $\mathbb{A}$와 $\mathbb{B}$에 대해

$|\mathbb{A}\times\mathbb{B}|^2 = |\mathbb{A}|^2|\mathbb{B}|^2 - |\mathbb{A}\cdot\mathbb{B}|^2$ 가 성립

왜냐면, Einstein summation을 이용해 증명할 때. ε_{ijk} 는 레비치비타.

$$|\mathbb{A}\times\mathbb{B}|^2 = (\mathbb{A}\times\mathbb{B})\cdot(\mathbb{A}\times\mathbb{B}) = \varepsilon_{ijk}(\mathbb{A}\times\mathbb{B})_i A_j B_k$$

$\mathbb{C}\cdot(\mathbb{A}\times\mathbb{B}) = \varepsilon_{ijk} C_i A_j B_k$

$\varepsilon_{ijk}\varepsilon_{iab} = \delta_j^a \delta_k^b$

$$= \varepsilon_{ijk}(\varepsilon_{iab} A_a B_b) A_j B_k$$

$$= A_j^2 B_k^2 - A_j A_k B_j B_k$$

$$= |\mathbb{A}|^2|\mathbb{B}|^2 - |\mathbb{A}\cdot\mathbb{B}|^2$$

$$\therefore |\mathbb{A}\times\mathbb{B}|^2 = |\mathbb{A}|^2|\mathbb{B}|^2 - |\mathbb{A}\cdot\mathbb{B}|^2$$

$$|\mathbb{A}\times(\mathbb{B}\times\mathbb{A})|^2 = |\mathbb{A}|^2|\mathbb{B}\times\mathbb{A}|^2 - |\cancelto{0}{\mathbb{A}\cdot(\mathbb{B}\times\mathbb{A})}|^2 = |\mathbb{A}|^2|\mathbb{B}\times\mathbb{A}|^2.$$

$$\therefore |\mathbb{A}\times(\mathbb{B}\times\mathbb{A})| = |\mathbb{A}||\mathbb{B}\times\mathbb{A}|.$$

$$\therefore \left|\frac{dT}{ds}\right| = \frac{|\mathbb{H}\times(\mathbb{K}\times\mathbb{H})|}{|\mathbb{H}|^4} = \frac{|\mathbb{H}||\mathbb{K}\times\mathbb{H}|}{|\mathbb{H}|^4} = \frac{|\mathbb{K}\times\mathbb{H}|}{|\mathbb{H}|^3} = \frac{|\mathbb{H}\times(\mathbb{H}\cdot\nabla)\mathbb{H}|}{|\mathbb{H}|^3}$$

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3번 문제.

$\phi(\mathbb{x})$는 스칼라장, $\mathbb{V}(\mathbb{x})$는 벡터장. 다음을 보여라.

$$(i)\quad \nabla\cdot(\phi\mathbb{V}) = (\nabla\phi)\cdot\mathbb{V} + \phi(\nabla\cdot\mathbb{V})$$

$$(ii)\quad \nabla\times(\phi\mathbb{V}) = (\nabla\phi)\times\mathbb{V} + \phi(\nabla\times\mathbb{V})$$

그리고 다음의 divergence 와 curl을 구하라.

$$r\mathbb{x},\quad \mathbb{a}(\mathbb{x}\cdot\mathbb{b}),\quad \mathbb{a}\times\mathbb{x},\quad \mathbb{x}/r^3.$$

역시 Einstein summation 으로 증명.

$$(i)\quad \nabla\cdot(\phi\mathbb{V}) = \frac{\partial}{\partial x_i}\phi v_i = \phi\frac{\partial}{\partial x_i}v_i + v_i\frac{\partial}{\partial x_i}\phi$$
$$= \phi\,\nabla\cdot\mathbb{V} + \mathbb{V}\cdot(\nabla\phi)$$

$$(ii)\quad \nabla\times(\phi\mathbb{V}) = \varepsilon_{ijk}\,\hat{i}\frac{\partial}{\partial x_j}\phi V_k = \varepsilon_{ijk}\,\hat{i}\,\phi\frac{\partial V_k}{\partial x_j} + \varepsilon_{ijk}\,\hat{i}\,V_k\frac{\partial\phi}{\partial x_j}$$
$$= \phi\left(\varepsilon_{ijk}\,\hat{i}\frac{\partial V_k}{\partial x_j}\right) + \varepsilon_{ijk}\,\hat{i}\left(\frac{\partial\phi}{\partial x_j}\right)V_k$$
$$= \phi(\nabla\times\mathbb{V}) + (\nabla\phi)\times\mathbb{V}$$

$$\boxed{\begin{aligned}&\nabla\cdot\mathbb{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \frac{\partial}{\partial x_i}V_i\\ &\nabla\times\mathbb{V} = \det\begin{pmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{pmatrix} = \varepsilon_{ijk}\,\hat{i}\frac{\partial}{\partial x_j}V_k\end{aligned}}$$

$r\mathbb{x}$ 계산. $(r=|\mathbb{x}|)$

$$\nabla\cdot(r\mathbb{x}) = r\,\nabla\cdot\mathbb{x} + \mathbb{x}\cdot(\nabla r)$$
$$= r + \frac{1}{r}\cdot r^2 = 2r$$
$$\nabla\times(r\mathbb{x}) = r(\nabla\times\mathbb{x}) + (\nabla r)\times\mathbb{x} = 0 + \frac{1}{r}\mathbb{x}\times\mathbb{x}$$
$$= 0$$

$$\nabla r = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{1}{r}\mathbb{x}$$
$$\nabla\cdot\mathbb{x} = 1$$
$$\nabla\times\mathbb{x} = 0$$

$\mathbb{a}(\mathbb{x}\cdot\mathbb{b})$ 계산

$$\nabla\cdot\{(\mathbb{x}\cdot\mathbb{b})\mathbb{a}\} = (\mathbb{x}\cdot\mathbb{b})\cancelto{0}{\nabla\cdot\mathbb{a}} + \mathbb{a}\cdot\{\nabla(\mathbb{x}\cdot\mathbb{b})\}$$
$$= \mathbb{a}\cdot\mathbb{b}$$
$$\nabla\times\{(\mathbb{x}\cdot\mathbb{b})\mathbb{a}\} = \{\nabla(\mathbb{x}\cdot\mathbb{b})\}\times\mathbb{a} + (\mathbb{x}\cdot\mathbb{b})\cancelto{0}{(\nabla\times\mathbb{a})}$$
$$= \mathbb{b}\times\mathbb{a}$$

$$\nabla(\mathbb{x}\cdot\mathbb{b}) = \mathbb{b}$$
$$\nabla\times\mathbb{a} = 0$$
$$\varepsilon_{\alpha\beta i}\,\varepsilon_{ijk} = \delta^\alpha_j\delta^\beta_k - \delta^\alpha_k\delta^\beta_j$$

$\mathbb{a}\times\mathbb{x}$ 계산

$$\nabla\cdot(\mathbb{a}\times\mathbb{x}) = \nabla\cdot(\varepsilon_{ijk}\,\hat{i}\,a_j x_k) = \varepsilon_{ijk}\frac{\partial}{\partial x_i}a_j x_k$$
$$= \varepsilon_{ijk}\,a_j\cancelto{0}{\frac{\partial}{\partial x_i}x_k} + \varepsilon_{ijk}\,x_k\cancelto{0}{\frac{\partial}{\partial x_i}a_j}$$
$$= 0$$

$$\nabla\times(\mathbb{a}\times\mathbb{x}) = \nabla\times(\varepsilon_{ijk}\,\hat{i}\,a_j x_k) = \varepsilon_{\alpha\beta i}\,\hat{\alpha}\frac{\partial}{\partial x_\beta}\varepsilon_{ijk}\,a_j x_k$$
$$= \varepsilon_{\alpha\beta i}\,\varepsilon_{ijk}\,\hat{\alpha}\frac{\partial}{\partial x_\beta}a_j x_k = \varepsilon_{\alpha\beta i}\,\varepsilon_{ijk}\,\hat{\alpha}\left(a_j\frac{\partial x_k}{\partial x_\beta} + x_k\frac{\partial a_j}{\partial x_\beta}\right)$$
$$= \hat{\alpha}(\delta^\alpha_j\delta^\beta_k - \delta^\alpha_k\delta^\beta_j)\left(a_j\frac{\partial x_k}{\partial x_\beta} + x_k\frac{\partial a_j}{\partial x_\beta}\right) = \left(a_j\frac{\partial x_k}{\partial x_k} + x_k\cancel{\frac{\partial a_j}{\partial x_k}}\right)\hat{j} - \left(a_j\cancel{\frac{\partial x_k}{\partial x_j}} + x_k\cancel{\frac{\partial a_j}{\partial x_j}}\right)\hat{k}$$
$$= \hat{j}\,a_j\underbrace{\frac{\partial x_k}{\partial x_k}}_{1} = \mathbb{a}$$

$\dfrac{\mathbb{X}}{r^3}$ 계산

$$\nabla\cdot(\mathbb{X}/r^3) = \frac{1}{r^3}(\nabla\cdot\mathbb{X}) + \mathbb{X}\cdot\left(\nabla\frac{1}{r^3}\right)$$
$$= \frac{1}{r^3} + \mathbb{X}\cdot\left(-\frac{3}{r^5}\mathbb{X}\right)$$
$$= \frac{1}{r^3} - \frac{3}{r^5}r^2 = -\frac{2}{r^3}$$

$$\nabla r^{-3} = \nabla(x^2+y^2+z^2)^{-\frac{3}{2}}$$
$$= -\frac{3}{2}\cdot r^{-5}(\hat{1}2x+\hat{2}2y+\hat{3}2z)$$
$$= -\frac{3}{r^5}\mathbb{X}$$

$$\nabla\times(\mathbb{X}/r^3) = (\nabla\times\mathbb{X})\frac{1}{r^3} + \mathbb{X}\times\left(\nabla\frac{1}{r^3}\right)$$
$$= 0 + \mathbb{X}\times\left(-\frac{3}{r^5}\mathbb{X}\right)$$
$$= 0$$

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4번 문제.

다음을 증명하라.

$$\nabla\times(\mathbf{u}\times\mathbf{v}) = \mathbf{u}(\nabla\cdot\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{u} - \mathbf{v}(\nabla\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{v}.$$

그리고 이것도 보여라.

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u})$$

역시 Einstein summation 을 이용한다.

$$\begin{aligned}
\nabla\times(\mathbf{u}\times\mathbf{v}) &= \nabla\times(\varepsilon_{ijk}\hat{i}\, u_j v_k) \\
&= \hat{\alpha}\,\varepsilon_{\alpha\beta i}\frac{\partial}{\partial x_\beta}\varepsilon_{ijk}u_j v_k \\
&= \varepsilon_{\alpha\beta i}\varepsilon_{ijk}\left(\frac{\partial}{\partial x_\beta}u_j v_k\right)\hat{\alpha} \\
&= \left(\delta^\alpha_j\delta^\beta_k - \delta^\alpha_k\delta^\beta_j\right)\left(\hat{\alpha}\frac{\partial}{\partial x_\beta}u_j v_k\right) \\
&= \hat{j}\frac{\partial}{\partial x_k}u_j v_k - \hat{k}\frac{\partial}{\partial x_j}u_j v_k \\
&= \hat{j}u_j\frac{\partial v_k}{\partial x_k} + \hat{j}v_k\frac{\partial u_j}{\partial x_k} - \hat{k}u_j\frac{\partial v_k}{\partial x_j} - \hat{k}v_k\frac{\partial u_j}{\partial x_j} \\
&= \mathbf{u}(\nabla\cdot\mathbf{v}) + (\mathbf{v}\cdot\nabla)\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{v} - \mathbf{v}(\nabla\cdot\mathbf{u})
\end{aligned}$$

$$\varepsilon_{\alpha\beta i}\varepsilon_{ijk} = \delta^\alpha_j\delta^\beta_k - \delta^\alpha_k\delta^\beta_j$$

왜냐? 자피 α와 β를 고르는 선택지가 $\begin{cases}\alpha=j \text{ and } \beta=k\\ \alpha=k \text{ and } \beta=j\end{cases}$ 둘 뿐이기 때문.

전자는 두 ε가 부호가 같은 반면 후자는 두 ε가 부호가 다르다.

증명 완.

$$\begin{aligned}
\mathbf{u}\times(\nabla\times\mathbf{u}) &= \nabla\left(\frac{1}{2}|\mathbf{u}|^2\right) - (\mathbf{u}\cdot\nabla)\mathbf{u} \\
&= \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}
\end{aligned}$$

$$\begin{aligned}
\mathbf{u}\times(\nabla\times\mathbf{u}) &= \mathbf{u}\times\left(\hat{i}\,\varepsilon_{ijk}\frac{\partial}{\partial x_j}u_k\right) \\
&= \hat{\alpha}\,\varepsilon_{\alpha\beta i}u_\beta\,\varepsilon_{ijk}\frac{\partial}{\partial x_j}u_k \\
&= \hat{\alpha}\,\varepsilon_{\alpha\beta i}\varepsilon_{ijk}u_\beta\frac{\partial}{\partial x_j}u_k \\
&= \hat{j}u_k\frac{\partial}{\partial x_j}u_k - \hat{k}\underbrace{u_j\frac{\partial}{\partial x_j}}_{\mathbf{u}\cdot\nabla}u_k \\
&= \hat{j}u_k\frac{\partial}{\partial x_j}u_k - (\mathbf{u}\cdot\nabla)\mathbf{u} \\
&= \hat{j}\frac{1}{2}\frac{\partial}{\partial x_j}u_k^2 - (\mathbf{u}\cdot\nabla)\mathbf{u} \\
&= \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}
\end{aligned}$$

증명완.

CM HW0

페이지 3
5번 문제.

$$\mathcal{U}(\mathbf{x}) = \left(e^x(x\cos y + \cos y - y\sin y),\ e^x(-x\sin y - \sin y - y\cos y),\ 0\right)$$

(i) 이것은 irrotational 하며, 어떤 scalar field ϕ 의 gradient 임을 보여라.

(ii) $\mathcal{U}$가 solenoidal 하며 어떤 vector field $\mathbb{V} = (0, 0, \psi)$ 의 curl 임을 보여라.

(i) 풀이

$$\underbrace{[\nabla\times\mathcal{U}]_x}_{\nabla\times\mathcal{U}\text{의 }x\text{성분}} = -\frac{\partial}{\partial z}e^x(-x\sin y - \sin y - y\cos y) = 0.$$

$$[\nabla\times\mathcal{U}]_y = \frac{\partial}{\partial z}e^x(x\cos y + \cos y - y\sin y) = 0.$$

$$\begin{aligned}
[\nabla\times\mathcal{U}]_z &= \frac{\partial}{\partial x}e^x(-x\sin y - \sin y - y\cos y) - \frac{\partial}{\partial y}e^x(x\cos y + \cos y - y\sin y) \\
&= e^x(-x\sin y - \sin y - y\cos y - \sin y) - e^x(-x\sin y - \sin y - \sin y - y\cos y) \\
&= e^x(-x\sin y - 2\sin y - y\cos y) - e^x(-x\sin y - 2\sin y - y\cos y) \\
&= 0.
\end{aligned}$$

$\therefore\ \nabla\times\mathcal{U} = 0.$

$$\mathcal{U} = \left(\frac{\partial\phi}{\partial x},\ \frac{\partial\phi}{\partial y},\ \frac{\partial\phi}{\partial z}\right)$$

$$\begin{aligned}
\phi &= \int e^x(x\cos y + \cos y - y\sin y)\,dx = \int e^x(-x\sin y - \sin y - y\cos y)\,dy \\
&= e^x(x-1)\cos y + e^x\cos y - e^x y\sin y \\
&= e^x(x\cos y - y\sin y)
\end{aligned}$$

$\therefore\ \phi = e^x(x\cos y - y\sin y)$

(ii) 풀이

$$\frac{\partial}{\partial x}u_x = e^x(x\cos y + \cos y - y\sin y + \cos y) = e^x(x\cos y + 2\cos y - y\sin y)$$

$$\frac{\partial}{\partial y}u_y = e^x(-x\cos y - \cos y - \cos y + y\sin y) = e^x(-x\cos y - 2\cos y + y\sin y)$$

$$\frac{\partial}{\partial z}u_z = 0$$

$$\therefore\ \nabla\cdot\mathcal{U} = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y + \frac{\partial}{\partial z}u_z = 0.$$

$$\mathcal{U} = \nabla\times\mathbb{V} = \left(\frac{\partial\psi}{\partial y},\ -\frac{\partial\psi}{\partial x},\ 0\right)$$

$$\begin{aligned}
\psi &= \int u_x\,dy = \int e^x(x\cos y + \cos y - y\sin y)\,dy = -\int u_y\,dx = \int e^x(x\sin y + \sin y + y\cos y)\,dx \\
&= e^x(x\sin y + \sin y + y\cos y - \sin y) \\
&= e^x(x\sin y + y\cos y)
\end{aligned}$$

$\therefore\ \psi = e^x(x\sin y + y\cos y)$

CM HW0

3페이지 6번문제

$\mathbb{F} = (3x^2\tan z - y^2 e^{-xy^2}\sin y,\ (\cos y - 2xy\sin y)e^{-xy^2},\ x^3\sec^2 z)$

이게 irrotational 함을 보이고, $\mathbb{F} = \nabla\phi$ 인 scalar potential $\phi(x)$ 를 찾아라.

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(-y^2 e^{-xy^2}\sin y\right) = -y^2\left(\frac{\partial}{\partial y}e^{-xy^2}\sin y\right) - 2ye^{-xy^2}\sin y$$

$$= -y^2\left(e^{-xy^2}\cos y - 2xye^{-xy^2}\sin y\right) - 2ye^{-xy^2}\sin y$$

$$= \underline{e^{-xy^2}\left(-y^2\cos y + 2xy^3\sin y - 2y\sin y\right)}$$

$$\frac{\partial F_x}{\partial z} = 3x^2\frac{1}{\cos^2 z} = \frac{3x^2}{\cos^2 z}$$

같다.

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}e^{-xy^2}(\cos y - 2xy\sin y) = -y^2 e^{-xy^2}(-2xy\sin y) - 2y\sin y\, e^{-xy^2} - (y^2\cos y)e^{-xy^2}$$

$$= \underline{e^{-xy^2}\left(2xy^3\sin y - 2y\sin y - y^2\cos y\right)}$$

$$\frac{\partial F_y}{\partial z} = 0$$

$$\frac{\partial F_z}{\partial x} = 3x^2\sec^2 z$$

$$\frac{\partial F_z}{\partial y} = 0$$

$$(\nabla\times\mathbb{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 0$$

$$(\nabla\times\mathbb{F})_y = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0$$

$$(\nabla\times\mathbb{F})_z = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 0$$

$\therefore\ \nabla\times\mathbb{F} = 0$

$$\mathbb{F} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\phi = \int 3x^2\tan z - y^2 e^{-xy^2}\sin y\, dx = x^3\tan z + e^{-xy^2}\sin y + C$$

$C=0$ 인게 좋다.

$$\phi = x^3\tan z + e^{-xy^2}\sin y$$

CM HW0

페이지 3

7번 문제.

3차원 vector field $\mathbb{F}$, $\nabla\cdot\mathbb{F}=0$ 이다.

$$\mathbb{A}(x)=\int_0^1 \mathbb{F}(t\mathbb{x})\times(t\mathbb{x})\,dt$$ 인 $\mathbb{A}$가 $\mathbb{F}=\nabla\times\mathbb{A}$

임을 보여라. 만약 모든 $\mathbb{R}^3$에서 $\mathbb{F}$가 정의되지 않으면 무슨 문제가 생기나?

$\mathbb{F}(t\mathbb{x})\times t\mathbb{x} = \hat{i}\,\varepsilon_{ijk}\,tF_j(t\mathbb{x})x_k$ 이건 아니고...

$$\nabla\times(\mathbb{u}\times\mathbb{v}) = \mathbb{u}(\nabla\cdot\mathbb{v})+(\mathbb{v}\cdot\nabla)\mathbb{u} - \mathbb{v}(\nabla\cdot\mathbb{u})-(\mathbb{u}\cdot\nabla)\mathbb{v}$$

$$\nabla\times\mathbb{A} = \nabla\times\left(\int_0^1 \mathbb{F}(t\mathbb{x})\times(t\mathbb{x})\,dt\right)$$

$$=\int_0^1 \nabla\times\{\mathbb{F}(t\mathbb{x})\times(t\mathbb{x})\}\,dt$$

$$=\int_0^1\{\mathbb{F}(t\mathbb{x})(\nabla\cdot(t\mathbb{x}))+((t\mathbb{x})\cdot\nabla)\mathbb{F}_{(t\mathbb{x})}-(t\mathbb{x})\cancel{(\nabla\cdot\mathbb{F}_{(t\mathbb{x})})}^{0}-(\mathbb{F}(t\mathbb{x})\cdot\nabla)(t\mathbb{x})\}\,dt$$

$\nabla\cdot(t\mathbb{x})=3t$, $(\mathbb{F}(t\mathbb{x})\cdot\nabla)(t\mathbb{x}) = t\mathbb{F}(t\mathbb{x})$.

$$\nabla\times\mathbb{A} = \int_0^1\{3t\,\mathbb{F}(t\mathbb{x})+((t\mathbb{x})\cdot\nabla)\mathbb{F}(t\mathbb{x})-t\mathbb{F}(t\mathbb{x})\}\,dt$$

$$=\int_0^1\{2t\,\mathbb{F}(t\mathbb{x})+((t\mathbb{x})\cdot\nabla)\mathbb{F}(t\mathbb{x})\}\,dt$$

$$\frac{d\mathbb{F}(t\mathbb{x})}{dt} = \frac{d(tx_i)}{dt}\frac{d\mathbb{F}(tx_i)}{d(tx_i)} = x_i\frac{d\mathbb{F}(tx_i)}{d(tx_i)}$$

$$((t\mathbb{x})\cdot\nabla)\mathbb{F}(t\mathbb{x}) = tx_i\frac{\partial\mathbb{F}(t\mathbb{x})}{\partial x_i} = t^2x_i\frac{\partial\mathbb{F}(t\mathbb{x})}{\partial(tx_i)} = t^2\frac{d\mathbb{F}(t\mathbb{x})}{dt}$$

$$=\int_0^1\{2t\,\mathbb{F}(t\mathbb{x})+t^2\frac{d\mathbb{F}}{dt}\}\,dt$$

$$=\int_0^1\frac{d}{dt}\{t^2\mathbb{F}\}\,dt$$

$$=\left[t^2\mathbb{F}_{(t\mathbb{x})}\right]_0^1$$

$$=\mathbb{F}(t\mathbb{x})$$

$\therefore$ $\nabla\times\mathbb{A}=\mathbb{F}$, when $\mathbb{A}(x)=\int_0^1\mathbb{F}(t\mathbb{x})\times(t\mathbb{x})\,dt$

모든 $\mathbb{R}^3$에서 $\mathbb{F}$가 정의되지 않으면 $\mathbb{A}$ 구하는 적분이 안 되지 않을까?

CM HW0

페이지 3 8번 문제

(u, v, w) 이 set of orthogonal curvilinear coordinates for $\mathbb{R}^3$ 이다.

$dV = h_u h_v h_w \, du \, dv \, dw$ 임을 보여라.

$dV = \rho \, d\rho \, d\phi \, dz$ 와 $dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$ 가 cylindrical 과 spherical coordinates 에서 성립함을 보여라.

h_u, h_v, h_w 가 뭔지 정의도 안 하고 증명하라 하면 곤란하다

h 들은 scaling factor 이며 미소 위치벡터 $d\mathbf{r}$ 에 대해

$$d\mathbf{r} = h_u du \hat{u} + h_v dv \hat{v} + h_w dw \hat{w}$$

를 만족하는 coordinates 마다 정해진 상수 이다.

$$h_u = \left|\frac{d\mathbf{r}}{du}\right|, \quad h_v = \left|\frac{d\mathbf{r}}{dv}\right|, \quad h_w = \left|\frac{d\mathbf{r}}{dw}\right|$$

$$\hat{u} = \widehat{\left(\frac{d\mathbf{r}}{du}\right)} = \frac{1}{h_u}\frac{d\mathbf{r}}{du}, \longrightarrow v \text{와 } w \text{에도 마찬가지}$$

dV 는 u, v, w 가 조금씩 변화된 미소변위 $h_u du \hat{u}, h_v dv \hat{v}, h_w dw \hat{w}$ 들이 만든 부피이다.

$$dV = (h_u du \hat{u}) \cdot (h_v dv \hat{v} \times h_w dw \hat{w})$$

직교성을 만족하므로, $\hat{v} \times \hat{w} = \hat{u}$

$$= (h_u du \hat{u}) \cdot (h_v h_w dv dw \hat{u})$$

$$= h_u h_v h_w \, du \, dv \, dw.$$

cylindrical coordinate 에서는, $\mathbf{r} = \rho\cos\phi \hat{x} + \rho\sin\phi \hat{y} + z\hat{z}$

$$h_\rho = \left|\frac{d\mathbf{r}}{d\rho}\right| = |\cos\phi \hat{x} + \sin\phi \hat{y}| = 1$$

$$h_\phi = \left|\frac{d\mathbf{r}}{d\phi}\right| = |-\rho\sin\phi \hat{x} + \rho\cos\phi \hat{y}| = \rho.$$

$$h_z = 1.$$

$$dV = h_\rho h_\phi h_z \, d\rho \, d\phi \, dz = \rho \, d\rho \, d\phi \, dz$$

spherical coordinates 에서는, $\mathbf{r} = r\cos\phi\sin\theta \hat{x} + r\sin\phi\sin\theta \hat{y} + r\cos\theta \hat{z}$

$$h_r = \left|\frac{d\mathbf{r}}{dr}\right| = |\cos\phi\sin\theta \hat{x} + \sin\phi\sin\theta \hat{y} + \cos\theta \hat{z}| = 1$$

$$h_\theta = \left|\frac{d\mathbf{r}}{d\theta}\right| = |r\cos\phi\cos\theta \hat{x} + r\sin\phi\cos\theta \hat{y} - r\sin\theta \hat{z}| = r$$

$$h_\phi = \left|\frac{d\mathbf{r}}{d\phi}\right| = |-r\sin\phi\sin\theta \hat{x} + r\cos\phi\sin\theta \hat{y}| = r\sin\theta.$$

$$dV = h_r h_\theta h_\phi \, dr \, d\theta \, d\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

ⓐ CM HW0.

페이지 4 9번 문제.

$\nabla(r^n) = n r^{n-2} \mathbf{x}$. $\nabla(\mathbf{a}\cdot\mathbf{x}) = \mathbf{a}$. $\mathbf{a}$는 constant vector.

이걸 증명하라

(i) Cartesian coordinates로 (ii) cylindrical polar coordinates로

(iii) spherical polar coordinates로 (IV) coordinate-independent approach로.

(i) cartesian coordinate,

$r = (x^2+y^2+z^2)^{\frac{1}{2}}$

$\nabla(r^n) = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{\frac{n}{2}}$

$= \frac{n}{2}(x^2+y^2+z^2)^{\frac{n-2}{2}}(2x\hat{x} + 2y\hat{y} + 2z\hat{z})$

$= n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x\hat{x} + y\hat{y} + z\hat{z})$

$= n r^{n-2}\mathbf{x}$.

$\nabla(\mathbf{a}\cdot\mathbf{x}) = \nabla(a_x x\hat{x} + a_y y\hat{y} + a_z z\hat{z})$

$= \left(\frac{\partial}{\partial x}a_x x\right)\hat{x} + \left(\frac{\partial}{\partial y}a_y y\right)\hat{y} + \left(\frac{\partial}{\partial z}a_z z\right)\hat{z}$

$= a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$

$= \mathbf{a}$

(ii) cylindrical polar coordinate, ρ, ϕ, z.

$\nabla = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$, $\mathbf{x} = \rho\hat{\rho} + z\hat{z}$

$r = (\rho^2+z^2)^{\frac{1}{2}}$

$\nabla(r^n) = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)(\rho^2+z^2)^{\frac{n}{2}}$

$= \frac{n}{2}(\rho^2+z^2)^{\frac{n-2}{2}}(2\rho\hat{\rho} + 2z\hat{z})$

$= n(\rho^2+z^2)^{\frac{n-2}{2}}(\rho\hat{\rho} + z\hat{z})$

$= n r^{n-2}\mathbf{x}$

$\mathbf{a} = a_\rho\hat{\rho} + a_z\hat{z}$, $\mathbf{a}\cdot\mathbf{x} = a_\rho\rho + a_z z$.

$\nabla(\mathbf{a}\cdot\mathbf{x}) = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)(a_\rho\rho + a_z z)$

$= a_\rho\hat{\rho} + a_z\hat{z}$

$= \mathbf{a}$

(iii) spherical polar coordinate, r, θ, ϕ.

$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}$, $\mathbf{x} = r\hat{r}$, $\mathbf{a} = a_r\hat{r}$

$\nabla(r^n) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)r^n = n r^{n-1}\hat{r} = n r^{n-2}(r\hat{r})$

$= n r^{n-2}\mathbf{x}$

$\nabla(\mathbf{a}\cdot\mathbf{x}) = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)(a_r r)$

$= a_r\hat{r} = \mathbf{a}$

(iv) coordinate independant approach.

어떤 scalar field $f(\mathbf{x})$ 에 대해, $\frac{df}{d\mathbf{x}} = \nabla f$. $\quad df = (\nabla f)\cdot d\mathbf{x}$.

$f(\mathbf{x}) = r^n$ 에 대해

$$df = \frac{df}{dr}\,dr = n r^{n-1}\,dr = n r^{n-1}\left(\frac{\mathbf{x}\cdot d\mathbf{x}}{r}\right) = (n r^{n-2}\,\mathbf{x})\cdot d\mathbf{x} = (\nabla f)\cdot d\mathbf{x}$$

$$r = \sqrt{\mathbf{x}\cdot\mathbf{x}}$$

$$\frac{dr}{d\mathbf{x}} = \frac{d}{d\mathbf{x}}\sqrt{\mathbf{x}\cdot\mathbf{x}} = \frac{1}{2\sqrt{\mathbf{x}\cdot\mathbf{x}}}(\mathbf{x}+\mathbf{x}) = \frac{\mathbf{x}}{\sqrt{\mathbf{x}\cdot\mathbf{x}}} = \frac{\mathbf{x}}{r}$$

$$r\,\cancel{dr = \mathbf{x}\cdot d}\mathbf{x}. \quad dr = \frac{1}{r}\,\mathbf{x}\cdot d\mathbf{x}$$

$\therefore\ \nabla f = n r^{n-2}\,\mathbf{x}$.

$f(\mathbf{x}) = \mathbf{a}\cdot\mathbf{x}$ 에 대해.

$$df = \left(\frac{df}{d\mathbf{x}}\right)\cdot d\mathbf{x} = \mathbf{a}\cdot d\mathbf{x} = (\nabla f)\cdot d\mathbf{x}.$$

$\therefore\ \nabla f = \mathbf{a}$

CM HW2
page 4, 10번 문제.

$$\mathbb{A}(x) = -\frac{1}{2}y\,e_x + \frac{1}{2}x\,e_y = \frac{1}{2}\rho\,e_\phi = \frac{1}{2}r\sin\theta\,e_\phi$$

$\nabla\times\mathbb{A}$ 를 각 coordinates 에서 계산하라.

$\nabla\times\mathbb{A}$ 의 일반식.
$$\nabla\times\mathbb{A} = h_i h_j h_k \begin{vmatrix} h_i e_i & h_j e_j & h_k e_k \\ \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} & \frac{\partial}{\partial x_k} \\ h_i A_i & h_j A_j & h_k A_k \end{vmatrix}$$

(i) in cartesian coordinates.

$$\nabla\times\mathbb{A} = \begin{vmatrix} e_x & e_y & e_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{1}{2}y & \frac{1}{2}x & 0 \end{vmatrix} = 1\,e_z.$$

(ii) in cylindrical coordinates.

$$\nabla\times\mathbb{A} = \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho\,e_\phi & e_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & \frac{1}{2}\rho^2 & 0. \end{vmatrix} = \frac{1}{\rho}\left(\frac{1}{2}\cdot 2\rho\right)e_z = e_z.$$

(iii) in spherical coordinates

$$\nabla\times\mathbb{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} e_r & r\,e_\theta & r\sin\theta\,e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & \frac{1}{2}r^2\sin^2\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left(r^2\sin\theta\cos\theta\,e_r - r^2\sin^2\theta\cdot e_\theta\right)$$

$$= \cos\theta\,e_r - \sin\theta\,e_\theta$$

$$[\nabla\times\mathbb{A}]_z = e_z\cdot(\cos\theta\,e_r - \sin\theta\,e_\theta)$$
$$= \cos\theta\,(e_z\cdot e_r) - \sin\theta\,(e_\theta\cdot e_z)$$
$$= \cos^2\theta + \sin^2\theta$$
$$= 1.$$

세 결과가 모두 같다.

CM HW0

페이지 4, 11번문제

Cylindrical polar coordinate.

$$\nabla = e_\rho \frac{\partial}{\partial \rho} + e_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + e_z \frac{\partial}{\partial z} \qquad \frac{\partial}{\partial \phi} e_\rho = \phi e_\phi \qquad \frac{\partial}{\partial \phi} e_\phi = - e_\rho$$

$\mathbb{A} = A_\rho e_\rho + A_\phi e_\phi + A_z e_z$ 일 때,

$\nabla \cdot \mathbb{A}$, $\nabla \times \mathbb{A}$, $\nabla^2 f$ 의 표현식을 구하라.

$$\nabla \circ \mathbb{A} = e_\rho \circ \left(\frac{\partial}{\partial \rho}\mathbb{A}\right) + \frac{1}{\rho} e_\phi \circ \left(\frac{\partial}{\partial \phi}\mathbb{A}\right) + e_z \circ \left(\frac{\partial}{\partial z}\mathbb{A}\right)$$

$$= \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\left(\frac{\partial A_\phi}{\partial \phi} + A_\rho\right) + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{\rho}\left(\rho \frac{\partial A_\rho}{\partial \rho} + A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\boxed{\begin{aligned} \frac{\partial}{\partial \rho}\mathbb{A} &= e_\rho \frac{\partial}{\partial \rho} A_\rho + e_\phi \frac{\partial}{\partial \rho} A_\phi + e_z \frac{\partial}{\partial \rho} A_z \\ \frac{\partial}{\partial \phi}\mathbb{A} &= A_\rho e_\phi + e_\rho \frac{\partial A_\rho}{\partial \phi} - A_\phi e_\rho + e_\phi \frac{\partial A_\phi}{\partial \phi} \\ &\quad e_z \frac{\partial A_z}{\partial \phi} \\ &= \left(\frac{\partial A_\rho}{\partial \phi} - A_\phi\right) e_\rho + \left(\frac{\partial A_\phi}{\partial \phi} + A_\rho\right) e_\phi \\ &\quad + \left(\frac{\partial A_z}{\partial \phi}\right) e_z \\ \frac{\partial}{\partial z}\mathbb{A} &= e_\rho \frac{\partial}{\partial z} A_\rho + e_\phi \frac{\partial}{\partial z} A_\phi + e_z \frac{\partial}{\partial z} A_z \end{aligned}}$$

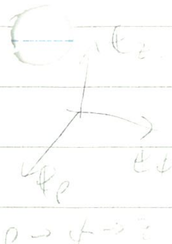
$$\nabla \times \mathbb{A} = e_\rho \times \left(\frac{\partial}{\partial \rho}\mathbb{A}\right) + \frac{1}{\rho} e_\phi \times \left(\frac{\partial}{\partial \phi}\mathbb{A}\right) + e_z \times \left(\frac{\partial}{\partial z}\mathbb{A}\right)$$

$$= (e_\rho \times e_\phi)\left(\frac{\partial}{\partial \rho}A_\phi\right) + (e_\rho \times e_z)\left(\frac{\partial}{\partial \rho}A_z\right)$$
$$+ \frac{1}{\rho}(e_\phi \times e_\rho)\left(\frac{\partial A_\rho}{\partial \phi} - A_\phi\right) + \frac{1}{\rho}(e_\phi \times e_z)\left(\frac{\partial}{\partial \phi}A_z\right)$$
$$+ (e_z \times e_\rho)\left(\frac{\partial}{\partial z}A_\rho\right) + (e_z \times e_\phi)\left(\frac{\partial}{\partial z}A_\phi\right)$$

$$= e_z\left(\frac{\partial}{\partial \rho}A_\phi\right) + e_\phi\left(-\frac{\partial}{\partial \rho}A_z\right) + e_z\left(-\frac{1}{\rho}\frac{\partial}{\partial \phi}A_\rho + \frac{1}{\rho}A_\phi\right)$$
$$+ e_\rho\left(\frac{1}{\rho}\frac{\partial}{\partial \phi}A_z\right) + e_\phi\left(\frac{\partial}{\partial z}A_\rho\right) + e_\rho\left(-\frac{\partial}{\partial z}A_\phi\right)$$

$$= e_\rho\left(\frac{1}{\rho}\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}A_\phi\right) + e_\phi\left(-\frac{\partial}{\partial \rho}A_z + \frac{\partial}{\partial z}A_\rho\right) + e_z\left(\frac{\partial}{\partial \rho}A_\phi + \frac{1}{\rho}A_\phi - \frac{1}{\rho}\frac{\partial}{\partial \phi}A_\rho\right)$$

$$= e_\rho\left(\frac{1}{\rho}\frac{\partial}{\partial \phi}A_z - \frac{\partial}{\partial z}A_\phi\right) + e_\phi\left(\frac{\partial}{\partial z}A_\rho - \frac{\partial}{\partial \rho}A_z\right) + \frac{1}{\rho}e_z\left\{\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial}{\partial \phi}A_\rho\right\}$$



$$\nabla^2 = \nabla \circ \nabla$$
$$= e_\rho \circ \left(\frac{\partial}{\partial \rho}\nabla\right) + \frac{1}{\rho} e_\phi \left(\frac{\partial}{\partial \phi}\nabla\right) + e_z\left(\frac{\partial}{\partial z}\nabla\right)$$
$$= \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\left\{\frac{\partial}{\partial \rho} + \frac{1}{\rho}\frac{\partial^2}{\partial \phi^2}\right\} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial}{\partial \rho}\nabla = e_\rho \frac{\partial}{\partial \rho^2} + e_\phi\left(-\frac{1}{\rho^2}\frac{\partial}{\partial \phi} + \frac{1}{\rho}\frac{\partial^2}{\partial \rho \partial \phi}\right) + e_z \frac{\partial^2}{\partial \rho \partial z}$$

$$\frac{\partial}{\partial \phi}\nabla = e_\rho \frac{\partial^2}{\partial \rho \partial \phi} + e_\phi \frac{\partial}{\partial \rho} + e_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi^2} - e_\rho \frac{1}{\rho}\frac{\partial}{\partial \phi}$$
$$+ e_z \frac{\partial^2}{\partial \phi \partial z}$$

$$\frac{\partial}{\partial z}\nabla = e_\rho \frac{\partial^2}{\partial z \partial \rho} + e_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi \partial z} + e_z \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 f = \frac{1}{\rho}\left(\rho \frac{\partial^2}{\partial \rho^2} f + \frac{\partial}{\partial \rho} f\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} f + \frac{\partial^2}{\partial z^2} f$$

* CM HW0
4페이지
12번 문제

Divergence theorem 을 $a \times \mathbb{A}$ 에 적용하라.

$$\int_V \nabla \times \mathbb{A}\, dV = \int_S d\vec{S} \times \mathbb{A}$$

임을 보여라. 이걸 $V = \{(x,y,z) : 0<x<a,\ 0<y<b,\ 0<z<c\}$ ~~에 적용하라.~~
$\mathbb{A}(x) = (z, 0, 0)$ 에 적용하여 진짜인지 확인하라.

풀이, 어떤 벡터장 $\mathbb{F}$에 대해 divergence theorem는

$$\int_V \nabla \cdot \mathbb{F}\, dV = \int_{\partial V} \mathbb{F} \cdot d\vec{S}$$

여기에 $\mathbb{F} = a \times \mathbb{A}$ 를 대입,

$$\int_V \nabla \cdot (a \times \mathbb{A})\, dV = \int_{\partial V} (a \times \mathbb{A}) \cdot d\vec{S}$$

를 풀어보자

◦ Einstein summation 을 사용

$$\begin{aligned}
\nabla \cdot (a \times \mathbb{A}) &= \nabla \cdot (e_i \varepsilon_{ijk} a_j A_k) \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_i} a_j A_k \\
&= \varepsilon_{ijk}\, a_j \frac{\partial}{\partial x_i} A_k \\
&= a_j [-\nabla \times \mathbb{A}]_j \\
&= -a \cdot (\nabla \times \mathbb{A})
\end{aligned}$$

$$\varepsilon_{ijk} \frac{\partial}{\partial x_i} A_k = [-\nabla \times \mathbb{A}]_j$$

$$\begin{aligned}
(a \times \mathbb{A}) \cdot d\vec{S} &= \varepsilon_{ijk}\, dS_i\, a_j A_k \\
&= a_j\, \varepsilon_{ijk}\, dS_i A_k \\
&= a_j [-d\vec{S} \times \mathbb{A}]_j \\
&= -a \cdot (d\vec{S} \times \mathbb{A})
\end{aligned}$$

$$\varepsilon_{ijk}\, dS_i A_k = [-d\vec{S} \times \mathbb{A}]_j$$

$$\therefore \int_V a \cdot (\nabla \times \mathbb{A})\, dV = \int_{\partial V} a \cdot (d\vec{S} \times \mathbb{A})$$

a가 constant 라면

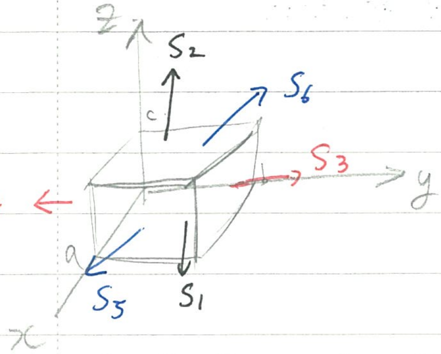
$$a \cdot \left[\int_V (\nabla \times \mathbb{A})\, dV\right] = a \cdot \left[\int_{\partial V} d\vec{S} \times \mathbb{A}\right]$$

$$\int_V \nabla \times \mathbb{A}\, dV = \int_{\partial V} d\vec{S} \times \mathbb{A}$$

$\mathbb{A}(x) = (z, 0, 0)$ 에 대해 풀자.

$\nabla \times \mathbb{A}(x) = (0, 1, 0)$.

$$\int_V \nabla \times \mathbb{A}\, dV = \int_V (0,1,0)\, dV = \int_0^c \int_0^b \int_0^a (0,1,0)\, dx\, dy\, dz = (0, abc, 0)$$



$\int d\vec{S} \times \mathbb{A}$ 를 풀기 위해

6개의 면에 이름 붙인다.

$S_1 : z = 0$. $\quad S_2 : z = c$

$S_3 : y = b \quad S_4 : y = 0$

$S_5 : x = a \quad S_6 : x = 0$.

$$\int_S d\vec{S} \times \mathbb{A} = \cancel{\int_{S_1} (0,0,-1) \times (\underset{0}{z},0,0)\, dS} + \int_{S_2} (0,0,1) \times (\underset{c}{z},0,0)\, dS$$

$$+ \int_{S_3} \underbrace{(0,1,0) \times (z,0,0)\, dS}_{\to 0} + \int_{S_4} (0,-1,0) \times (z,0,0)\, dS$$

$$+ \int_{S_5} (1,0,0) \times (z,0,0)\, dS + \int_{S_6} \underbrace{\cancel{(-1,0,0) \times (z,0,0)}}_{\to 0}\, dS$$

$$= \int_0^b \int_0^a (0, c, 0)\, dx\, dy + \int_0^a \int_0^c (0, 0, -z)\, dz\, dx$$

$$+ \int_0^a \int_0^c (0, 0, z)\, dz\, dx$$

$$= (0, abc, 0) + (0, 0, -\tfrac{1}{2}c^2 a) + (0, 0, \tfrac{1}{2}c^2 a)$$

$$= (0, abc, 0).$$

두 가지 풀이 모두 같은 값이 나옴.

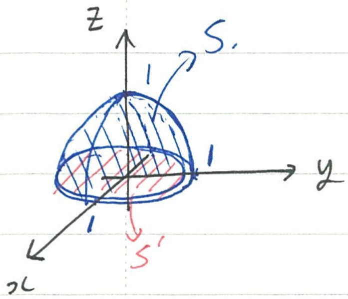
CM HW0

페이지 4, 13번 문제

$$\mathbb{F}(x) = (x^3+3y+z^2,\ y^3,\ x^2+y^2+3z^2).$$

$$S : \quad 1-z = x^2+y^2 \qquad 0 \le z \le 1$$

divergence theorem과 cylindrical polar coordinates 를 이용해 $\int_S \mathbb{F}\cdot d\vec{S}$ 를 풀어라. $d\vec{S} = (2\rho\cos\phi, 2\rho\sin\phi, 1)\rho\, d\rho\, d\phi$ 로 먼저 보여야 할 것이다.



S'는 S와 더해져 닫힌 곡면을 만드는 영역

$S' : z=0 , \ 0 < x^2+y^2 < 1.$

S+S' 내부 공간을 V 라 부른다.

divergence theorem 에 따라,

$$\underbrace{\int_S \mathbb{F}\cdot d\vec{S} + \int_{S'} \mathbb{F}\cdot d\vec{S'}}_{} = \int_V \nabla\cdot\mathbb{F}\, dV.$$

↳ 이 둘을 먼저 구하자.

$$\nabla\cdot\mathbb{F} = \left\{\frac{\partial}{\partial x}(x^3+3y+z^2)\right\} + \left\{\frac{\partial}{\partial y} y^3\right\} + \left\{\frac{\partial}{\partial z}(x^2+y^2+3z^2)\right\}$$
$$= 3x^2+3y^2+6z$$
$$= 3\rho^2+6z,$$

$$\int_V \nabla\cdot\mathbb{F}\, dV = \int_0^{2\pi}\int_0^1\int_0^{1-\rho^2}(3\rho^2+6z)\rho\, dz\, d\rho\, d\phi$$

$$\int_0^{1-\rho^2}(3\rho^2+6z)\rho\, dz = \left[3\rho^3 z + 3\rho z^2\right]_0^{1-\rho^2} = 3\rho^3(1-\rho^2)+3\rho(1-\rho^2)^2 = -3\rho^3+3\rho$$

$$= \int_0^{2\pi}\int_0^1(-3\rho^3+3\rho)\, d\rho\, d\phi$$
$$= 2\pi\left(-\frac{3}{4}+\frac{3}{2}\right)$$
$$= \frac{3}{2}\pi$$

$F_z = x^2+y^2+3z^2 = \rho^2+3z.$ S'가 법선벡터가 -z 방향이기에

$$\int_{S'} \mathbb{F}\cdot d\vec{S'} = -\int_0^1\int_0^{2\pi} F_z\, \rho\, d\phi\, d\rho \qquad z=0.$$
$$= -\int_0^1\int_0^{2\pi}\rho^3\, d\phi\, d\rho$$
$$= -2\pi\cdot\frac{1}{4} = -\frac{1}{2}\pi$$

아마. $\int_S \mathbb{F}\cdot d\vec{S} = \frac{3}{2}\pi + \frac{1}{2}\pi = 2\pi.$ 일 것이다.

이것을 직접구하면, 먼저 S의 normal vector를 구해야 한다.

곡면의 좌표는 $(x, y, 1-x^2-y^2)$, x와 y에 대한 tangent vector를 구하면

$$\vec{T}_x = \frac{\partial}{\partial x}(x, y, 1-x^2-y^2) = (1, 0, -2x)$$
$$\vec{T}_y = \frac{\partial}{\partial y}(x, y, 1-x^2-y^2) = (0, 1, -2y)$$

$$\mathbb{F}(\mathbf{x}) = (x^3+3y+z^2,\ y^3,\ x^2+y^2+3z^2)$$

$$\vec{T}_x = (1, 0, -2x), \quad \vec{T}_y = (0, 1, -2y)$$

$$d\vec{S} = \vec{T}_x \times \vec{T}_y \, dx\,dy$$
$$= (2x, 2y, 1)\, dx\,dy$$
$$= (2\rho\cos\phi,\ 2\rho\sin\phi,\ 1)\, \rho\, d\rho\, d\phi$$

$$\mathbb{F}_x = x^3+3y+z^2 = \rho^3\cos^3\phi + 3\rho\sin\phi + \rho^4 - 2\rho^2 + 1$$
$$F_y = \rho^3 \sin^3\phi$$
$$F_z = \rho^2 + 3\rho^4 - 6\rho^2 + 3 = 3\rho^4 - 5\rho^2 + 3$$

$$\int_S \mathbb{F}\cdot d\vec{S} = \int_0^{2\pi}\int_0^1 \big(2\rho^4\cos^4\phi + 6\rho^2\sin\phi\cos\phi + 2\rho^5\cos\phi - 4\rho^3\cos\phi + 2\rho\cos\phi$$
$$+ 2\rho^4\sin^4\phi + 3\rho^4 - 5\rho^2 + 3\big)\,\rho\, d\rho\, d\phi$$

$$= \int_0^{2\pi}\Big(\frac{1}{3}\cos^4\phi + \frac{3}{2}\sin\phi\cos\phi + \frac{2}{7}\cos\phi - \frac{4}{5}\cos\phi + \frac{2}{3}\cos\phi$$
$$+ \frac{1}{3}\sin^4\phi + \frac{1}{2} - \frac{5}{4} + \frac{3}{2}\Big)\, d\phi$$
$$= \int_0^{2\pi}\Big(\frac{1}{3}\cos^4\phi + \frac{1}{3}\sin^4\phi + \frac{3}{4}\Big)\, d\phi$$

$$= \frac{1}{3}\times 2 \times \frac{3}{4}\pi + \frac{3}{4}\times 2\pi = \left(\frac{1}{2}+\frac{3}{2}\right)\pi = 2\pi$$

와!

두 방법 모두 같은 값 나온다.

CM HW0

5페이지 1번문제

$$I = \oint_C -x^2 y\,dx + x y^2\,dy.$$

이 적분을

(i) C 가 중심이 원점이고 반지름이 R 인 반시계 방향 경로 일때, Green's theorem 을 이용하라. R = b 와 R = a 일 때의 결과를 비교하라.

(ii) ~~앞변~~ 변의 길이가 l 이고 중심이 원점인 정사각형 일 때. 정사각형이 회전해도 적분값은 바뀌지 않음을 보여라.

$\mathbb{F} = (-x^2 y,\ x y^2,\ 0)$

polar coordinate 로 표현, r 과 ϕ 를 이용.

$$\begin{pmatrix}\hat{x}\\ \hat{y}\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\phi}\end{pmatrix} \qquad \begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\mathbb{F} = -x^2 y\,\hat{x} + x y^2\,\hat{y} = -r^3\cos^2\phi\sin\phi\,(\cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}) + r^3\cos\phi\sin^2\phi\,(\sin\phi\,\hat{r} + \cos\phi$$

$$= r^3\cos\phi\sin\phi\,(-\cos^2\phi + \sin^2\phi)\,\hat{r} + 2r^3\cos^2\phi\sin^2\phi\,\hat{\phi}$$

(i) $$I = \int_0^{2\pi} (\mathbb{F}\cdot\hat{\phi})\, r\,d\phi \Big|_{r=R} = 2R^4\int_0^{2\pi}\cos^2\phi\sin^2\phi\,d\phi = 2R^4\int_0^{2\pi}\frac{1}{4}\sin^2(2\phi)\,d\phi$$

$\cos\phi\sin\phi = \frac{1}{2}\sin(2\phi)$ $$= 2R^4 \times \frac{1}{4} \times \pi = \frac{\pi}{2}R^4$$

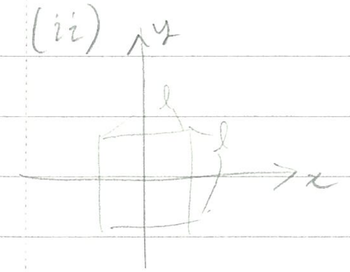
Green's theorem 로 이용해 보자. $|\nabla\times\mathbb{F}| = y^2 + x^2 = r^2$

$$I = \int_0^{2\pi}\int_0^R |\nabla\times\mathbb{F}|\, r\,dr\,d\phi = \int_0^{2\pi}\int_0^R r^3\,dr\,d\phi = 2\pi\cdot\frac{1}{4}R^4 = \frac{\pi}{2}R^4.$$

값이 같다.

$R = a$, $R = b$ 일 때는 $I_a = \frac{\pi}{2}a^4$ $\quad I_b = \frac{\pi}{2}b^4$

(ii)



$$I = \int_{-\frac{l}{2}}^{\frac{l}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}} |\nabla\times\mathbb{F}|\,dx\,dy = \int_{-\frac{l}{2}}^{\frac{l}{2}}\int_{-\frac{l}{2}}^{\frac{l}{2}} (x^2 + y^2)\,dx\,dy$$

$$= \int_{-\frac{l}{2}}^{\frac{l}{2}} \left(\frac{l^3}{12} + l y^2\right) dy = \left[\frac{l^3}{12}y + \frac{l}{3}y^3\right]_{-\frac{l}{2}}^{\frac{l}{2}}$$

$$= \frac{1}{12}l^4 + \frac{1}{3}l \times 2 \times \left(\frac{l}{2}\right)^3 = \frac{1}{12}l^4 + \frac{1}{12}l^4 = \frac{1}{6}l^4$$

사각형을 돌리는 건 귀찮으니, 좌표계를 돌리자,

$$\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix}, \qquad |\nabla\times\mathbb{F}| = x^2 + y^2 = (x'\cos\phi - y'\sin\phi)^2 + (x'\sin\phi + y'\cos\phi)^2$$

$$= (x')^2 + (y')^2$$

여전히 $|\nabla\times\mathbb{F}|$가 같으므로, I도 같을 것이다.

CM HW0

5 페이지 2번 문제.

$\mathbb{F}(\mathbb{x}) = (y, -x, z)$ $S = \{x^2+y^2+z^2=1,\ z \geq 0\}$

Stokes' theorem 을 증명하라.

미소 면적벡터, spherical coordinate 에서.

$d\vec{s} = \hat{r}\, r\sin\theta\, d\phi d\theta \underset{r=1}{=} \hat{r}\sin\theta\, d\phi d\theta.$

$$\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix} = \begin{pmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi\\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi\\ \cos\theta & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}\hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{pmatrix}$$

$\nabla\times\mathbb{F} = (0, 0, -2) = -2\hat{z} = -2\cos\theta\,\hat{r} + 2\sin\theta\,\hat{\theta}$

$$\int_S (\nabla\times\mathbb{F})\cdot d\vec{s} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} -2\cos\theta\sin\theta\, d\theta d\phi = -4\pi\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\theta)\,d\theta$$
$$= -4\pi\left[-\frac{1}{4}\cos(2\theta)\right]_0^{\frac{\pi}{2}}$$
$$= -2\pi.$$

$x^2+y^2=1$ 인 반시계 방향 경로에 대해 선적분

$[\mathbb{F}]_\phi = -y\sin\phi - x\cos\phi = -r\sin\theta\sin^2\phi - r\sin\theta\cos^2\phi = -r\sin\theta.$

$$\int_C \mathbb{F}\cdot d\vec{l} \underset{r=1}{=} \int_0^{2\pi}[\mathbb{F}]_\phi\, d\phi = \int_0^{2\pi} -\sin\theta\, d\phi \underset{\theta=\frac{\pi}{2}}{=} -2\pi.$$

두 방법 모두 값이 -2π 로 같다.

CM HW0

5페이지 3번 문제

$\mathbb{A} = \mathbb{a} \times \mathbb{F}$ 에 Stokes' theorem을 적용하여

$$\oint_c d\mathbb{X} \times \mathbb{F} = \int_s (d\vec{s} \times \nabla) \times \mathbb{F}.$$

임을 보여라. $(0,0,0)$과 $(1,1,0)$을 반대점으로 가지는 정사각형 경로와 $\mathbb{F}(\mathbb{X}) = \mathbb{X}$로 증명하라.

$$\oint_c (\mathbb{a} \times \mathbb{F}) \cdot d\vec{l} = \int_s \nabla \times (\mathbb{a} \times \mathbb{F}) \cdot d\vec{s} \iff \oint_c d\vec{l} \times \mathbb{F} = \int_s (d\vec{s} \times \nabla) \times \mathbb{F}$$

임을 보여야 한다.

$$(\mathbb{a} \times \mathbb{F}) \cdot d\vec{l} = (\hat{i}\, \varepsilon_{ijk} a_j F_k) \cdot d\vec{l} = \varepsilon_{ijk}\, l_i a_j F_k = a_j \underbrace{\varepsilon_{ijk}\, l_i F_k}_{[-d\vec{l} \times \mathbb{F}]_j}$$

$$= \mathbb{a} \cdot [-d\vec{l} \times \mathbb{F}]$$

$$\nabla \times (\mathbb{a} \times \mathbb{F}) = \nabla \times (\hat{i}\, \varepsilon_{ijk} a_j F_k) = \varepsilon_{mli}\, \hat{m} \frac{\partial}{\partial x_l} \varepsilon_{ijk} a_j F_k$$

$$= (\delta_{mj}\delta_{lk} - \delta_{mk}\delta_{lj})\, \hat{m} \frac{\partial}{\partial x_l} a_j F_k$$

$$= \hat{j} \frac{\partial}{\partial x_k} a_j F_k - \hat{k} \frac{\partial}{\partial x_j} a_j F_k.$$

$$\nabla \times (\mathbb{a} \times \mathbb{F}) \cdot d\vec{s} = dS_j \frac{\partial}{\partial x_k} a_j F_k - dS_k \frac{\partial}{\partial x_j} a_j F_k$$

$$= a_j \left[dS_j \frac{\partial}{\partial x_k} F_k - dS_k \frac{\partial}{\partial x_j} F_k \right]$$

$$= -\mathbb{a} \cdot \left(-\hat{j}\, dS_j \frac{\partial}{\partial x_k} F_k + \hat{j}\, dS_k \frac{\partial}{\partial x_j} F_k \right)$$

$$= -\mathbb{a} \cdot \left\{ (\delta_{mj}\delta_{lk} - \delta_{mk}\delta_{lj}) (\hat{l}\, dS_j \frac{\partial}{\partial x_k} F_m) \right\}$$

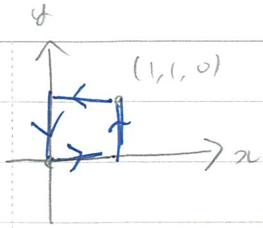
$$= -\mathbb{a} \cdot \left\{ \varepsilon_{lim} \varepsilon_{ijk}\, \hat{l}\, dS_j \frac{\partial}{\partial x_k} F_m \right\}$$

$$= -\mathbb{a} \cdot \left\{ (d\vec{s} \times \nabla) \times \mathbb{F} \right\}$$

$$\therefore \oint_c \mathbb{a} \cdot [-d\vec{l} \times \mathbb{F}] = \int_s \mathbb{a} \cdot [-(d\vec{s} \times \nabla) \times \mathbb{F}]$$

$$\mathbb{a} \cdot \oint_c -d\vec{l} \times \mathbb{F} = \mathbb{a} \cdot \int_s -(d\vec{s} \times \nabla) \times \mathbb{F}$$

$$\oint_c d\vec{l} \times \mathbb{F} = \int_s (d\vec{s} \times \nabla) \times \mathbb{F}$$



$$\mathbb{F}(\mathbb{x}) = x\hat{x} + y\hat{y} + z\hat{z}$$

$$d\mathbb{x} \times \mathbb{F} = (z\,dy - y\,dz)\hat{x} + (x\,dz - z\,dx)\hat{y} + (y\,dx - x\,dy)\hat{z}$$

$$\oint_C d\mathbb{x} \times \mathbb{F} = \cancelto{0}{\left(\oint_C z\,dy - y\,dz\right)\hat{x}}$$

$$+ \cancel{\left(\oint_C x\,dz - z\,dx\right)\hat{y}}$$

$$+ \left(\oint_C y\,dx - x\,dy\right)\hat{z}$$

$$= (-1 - 1)\hat{z}$$

$$= -2\hat{z}$$

$$d\vec{S} = \hat{z}\,dx\,dy.$$

$$(d\vec{S} \times \nabla) = \left(-dx\,dy\,\frac{\partial}{\partial y}\right)\hat{x} + \left(dx\,dy\,\frac{\partial}{\partial x}\right)\hat{y}$$

$$(d\vec{S} \times \nabla) \times \mathbb{F} = +dx\,dy\left\{\left(-\frac{\partial}{\partial y}y - \frac{\partial}{\partial x}x\right)\hat{z}\right\} = -2\,dx\,dy\,\hat{z}$$

$$\int_S (d\vec{S} \times \nabla) \times \mathbb{F} = -2\hat{z}\int_0^1\int_0^1 dx\,dy = -2\hat{z}.$$

CM HW0

5페이지 4번문제

$$S = \{\mathbf{x} : |\mathbf{x}| = 1\} \qquad \mathbf{F}(\mathbf{x}) = \frac{\mathbf{x}}{r^3}$$

$\int_S \mathbf{F}\cdot d\vec{S}$ 를 계산하라. $\mathbf{F} = \nabla\times\mathbf{A}$ 를 만족하는 $\mathbf{A}$ 가 없음을 보여라.

$\mathbf{F}$ 는 정전하에 의한 전기포텐셜이다. $\nabla\cdot\mathbf{F}$ 는 디랙 델타가 나오는 것으로 유명하다. 직접 $\nabla\cdot\mathbf{F}$ 를 계산하면,

$\mathbf{F}(\mathbf{x}) = \frac{1}{r^2}\hat{r}$ in spherical coordinate

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r} r^2\sin\theta\, F_r + \frac{\partial}{\partial\theta} r\sin\theta\, F_\theta + \frac{\partial}{\partial\phi} r F_\phi\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \cdot \frac{1}{r^2}$$

$$= 0 \quad ???$$

그러나... 이 식으로는 $r=0$ 에서 $\nabla\cdot\mathbf{F}$ 가 얼마인지 알 수 없다.

Divergence theorem 을 이용해 계산하면, $d\vec{S} = \hat{r}\, r^2\sin\theta\, d\phi\, d\theta$.

$$\int_S \mathbf{F}\cdot d\vec{S} = \int_0^{\pi}\int_0^{2\pi}\left(\frac{1}{r^2}\hat{r}\right)\cdot\hat{r}\; r^2\sin\theta\, d\phi\, d\theta \,\Big|_{r=1}$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin\theta\, d\phi\, d\theta.$$

$$= 4\pi.$$

즉, $\nabla\cdot\mathbf{F}(\mathbf{x}) = 4\pi\,\delta(\mathbf{x})$ 임을 알 수 있다

만약 어떤 $\mathbf{A}$ 에 대해 $\mathbf{F} = \nabla\times\mathbf{A}$ 라면, $\nabla\cdot\mathbf{F} = \nabla\cdot(\nabla\times\mathbf{A}) = 0$ 이어야만 한다.

증명 $\rightarrow$ $\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\left(\epsilon_{ijk}\,\hat{i}\,\frac{\partial}{\partial x_j}A_k\right)$

$$= \epsilon_{ijk}\frac{\partial^2}{\partial x_i\,\partial x_j}A_k$$

$\frac{\partial^2}{\partial x_i\,\partial x_j} = \frac{\partial^2}{\partial x_j\,\partial x_i}$ 이므로,

$$= 0$$

따라서 $\nabla\cdot\mathbf{F} \neq 0$ 이므로, $\mathbf{A}$ 는 존재하지 않는다.

CM HW0

5페이지 5번 문제

$$\mathbb{A}(\mathbb{x}) = \frac{1}{(x^2+y^2)r}(yz, -xz, 0)$$

$\nabla \times \mathbb{A}$ 를 계산하라. surface

$$S_\varepsilon = \{\mathbb{x} : |\mathbb{x}| = 1,\ x^2+y^2 \geq \varepsilon^2\}$$

에 대해 stokes theorem 을 적용하라. 앞선 문제의 결론과 모순되는가 아닌가

일단 spherical coordinate로 변환한다.

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta. \\ x^2+y^2 = r^2\sin^2\theta. \end{cases} \qquad \begin{cases} \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}. \end{cases}$$

$$\mathbb{A}(\mathbb{x}) = \frac{1}{r^3\sin^2\theta}\left(yz\,\hat{x} - xz\,\hat{y}\right)$$

$$\begin{aligned} yz\,\hat{x} - xz\,\hat{y} &= r^2\sin\theta\cos\theta\sin\phi\,\hat{x} - r^2\sin\theta\cos\theta\cos\phi\,\hat{y} \\ &= r^2\sin\theta\cos\theta(\sin\phi\,\hat{x} - \cos\phi\,\hat{y}) \\ &= r^2\sin\theta\cos\theta(-\sin^2\phi\,\hat{\phi} - \cos^2\phi\,\hat{\phi}) \\ &= -r^2\sin\theta\cos\theta\,\hat{\phi} \end{aligned}$$

$$\mathbb{A}(\mathbb{x}) = -\frac{1}{r}\frac{\cos\theta}{\sin\theta}\hat{\phi}$$

$$\nabla\times\mathbb{A}(\mathbb{x}) = \hat{r}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(r\sin\theta A_\phi) = \hat{r}\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(-\cos\theta) = \frac{1}{r^2}\hat{r}$$

와! 앞선 문제의 $\mathbb{F}(\mathbb{x})$ 가 나온다.

이제 stokes theorem 을 적용시킬 차례

$$d\vec{S} = \hat{r}\,r^2\sin\theta\,d\phi\,d\theta \underset{r=1}{\underbrace{=}} \hat{r}\sin\theta\,d\phi\,d\theta$$

$$\sin^{-1}\varepsilon < \theta < \pi - \sin^{-1}\varepsilon.$$

$$\int_{\sin^{-1}\varepsilon}^{\pi-\sin^{-1}\varepsilon}\int_0^{2\pi}\sin\theta\,d\phi\,d\theta = 2\pi\Big[-\cos\theta\Big]_{\sin^{-1}\varepsilon}^{\pi-\sin^{-1}\varepsilon}$$

우리가 바라는 건 S_ε가 구가 되는 상황, $\varepsilon\to 0$ 의 극한이다.

$$\lim_{\varepsilon\to 0}\int_{S_\varepsilon}(\nabla\times\mathbb{A})\cdot d\vec{S} = 2\pi\lim_{\varepsilon\to 0}\Big[-\cos\theta\Big]_{\sin^{-1}\varepsilon}^{\pi-\sin^{-1}\varepsilon} = 4\pi.$$

그러면 점전하 전기장의 벡터 포텐셜이 있는 것인가? $\mathbb{A}$가 우리가 찾던 그것인가?

그러나 $\mathbb{A}$는 단위구 표면 전체에서 정의되지 않는다. (0,0,±1) 에서 정의 안됨

그러나 A는 단위구 표면에서 정의되지 않는다. (0,0,1)과 (0,0,-1)에서 정의되지 않는다.