AM HW4 20262182 김신지

4. Let $g_{\mu\nu}$ be a rank $(0,2)$ tensor. In a basis, one can regard the components $g_{\mu\nu}$ as elements of an $n \times n$ matrix, so that one may define the determinant $g = \det(g_{\mu\nu})$. How does g transform under a change of basis?

G 는 $g_{\mu\nu}$ 를 나타내는 $n \times n$ matrix

Basis transform matrix 가 T 일 때,

바 꾼 basis 에서 $g'_{\mu\nu}$ 는 G'

$$G' = T^{-1} G T$$

$$\det(G^{-1}) = \det(T^{-1}) \det(G) \det(T)$$

역행렬의 특성 상, $\det(T^{-1}) = \frac{1}{\det(T)}$

$$\det(G^{-1}) = \frac{1}{\det(T)} \det(G) \det(T)$$

$$= \det(G)$$

$$= g$$

즉, g 는 basis transform 에 invariant 하다.

5*. Use the Leibniz rule to derive the formula for the Lie derivative of a 1-form ω, valid in any coordinate basis:

$$(\mathcal{L}_X\omega)_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$$

[Hint: consider $(\mathcal{L}_X\omega)(Y)$ for a vector field Y.] Show that the Lie derivative of a $(0,2)$ tensor g is

$$(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho$$

For a p-form η, define $\iota_X\eta$ to be the $(p-1)$-form that results from contracting a vector field X with the first index of η. Show that for a 1-form ω,

$$\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega)$$

$(\mathcal{L}_X\omega)_\mu = X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu$ 를 증명해 보자.

$(\mathcal{L}_X\omega)(Y)$ 를 이용, $Y = \partial_\mu$ 를 대입. (∂_μ는 basis vector)

$(\mathcal{L}_X\omega)(\partial_\mu) = (\mathcal{L}_X\omega)_\mu$ 으로 쉽게 계산

ω를 명시적으로 나타내면, $\omega = \omega_\mu dx^\mu$ (Einstein summation notation 이용)

X를 명시적으로 나타내면, $X = X^\nu\partial_\nu$

$\omega(\partial_\mu)$ 전체에 $\mathcal{L}_X$를 적용 뒤 Leibniz rule 을 적용

$$\mathcal{L}_X\,\omega(\partial_\mu) = (\mathcal{L}_X\omega)(\partial_\mu) + \omega(\mathcal{L}_X\partial_\mu)$$

두 벡터장 X,Y 에 대해
$\mathcal{L}_X Y = [X,Y]$

$$\mathcal{L}_X\omega_\mu = (\mathcal{L}_X\omega)_\mu + \omega([X,\partial_\mu])$$

ω_μ는 0-form

$$X(\omega_\mu) = (\mathcal{L}_X\omega)_\mu + \omega(X^\nu\partial_\nu\partial_\mu - \partial_\mu X^\nu\partial_\nu)$$

$$X^\nu\partial_\nu\omega_\mu = (\mathcal{L}_X\omega)_\mu + \omega_\alpha[X^\nu\partial_\nu\partial_\mu - \partial_\mu X^\nu\partial_\nu]^\alpha$$

$$X^\nu\partial_\nu\omega_\mu = (\mathcal{L}_X\omega)_\mu - \omega_\alpha\partial_\mu X^\alpha$$

$[X^\nu\partial_\nu\partial_\mu - \partial_\mu X^\nu\partial_\nu]^\alpha$ 를 구해 보자. smooth 0-form, f를 넣으면,

$$\begin{aligned} X^\nu\partial_\nu\partial_\mu f - \partial_\mu X^\nu\partial_\nu f &= X^\nu\partial_\nu\partial_\mu f - (\partial_\mu X^\nu)\partial_\nu f - X^\nu(\partial_\mu\partial_\nu f) \\ &= X^\nu\partial_\mu\partial_\nu f - X^\nu\partial_\mu\partial_\nu f - (\partial_\mu X^\nu)\partial_\nu f \\ &= -(\partial_\mu X^\nu)\partial_\nu f \end{aligned}$$

$$\therefore [X^\nu\partial_\nu\partial_\mu - \partial_\mu X^\nu\partial_\nu]^\alpha = -\partial_\mu X^\alpha$$

index를 바꾸어 정리

나중을 위해 정리,

$$\mathcal{L}_X\partial_\mu = -\partial_\mu X^\alpha\partial_\alpha$$

$$(\mathcal{L}_X\omega)_\mu = X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu$$

이어서

g가 (0,2) tensor 일때, $(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho$ 증명

임의의 벡터장 Y, Z 에 대해,

$\mathcal{L}_X g(Y,Z)$ 에 Leibniz rule 을 적용하면,

$$\mathcal{L}_X g(Y,Z) = (\mathcal{L}_X g)(Y,Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z)$$

g가 (0,2) tensor 일때, $(\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho$ 증명.

임의의 벡터장 Y, Z 에 대해,

$$\mathcal{L}_X(g(Y,Z)) = (\mathcal{L}_X g)(Y,Z) + g(\mathcal{L}_X Y, Z) + g(Y, \mathcal{L}_X Z)$$

$Y = \partial_\mu$, $Z = \partial_\nu$ 대입, $(\mathcal{L}_X g)(\partial_\mu, \partial_\nu) = (\mathcal{L}_X g)_{\mu\nu}$

$$(\mathcal{L}_X g)_{\mu\nu} = \mathcal{L}_X(g(\partial_\mu,\partial_\nu)) - g(\mathcal{L}_X\partial_\mu, \partial_\nu) - g(\partial_\mu, \mathcal{L}_X\partial_\nu)$$

$\mathcal{L}_X \partial_\mu = -\partial_\mu X^\rho \partial_\rho$

$$= \mathcal{L}_X(g_{\mu\nu}) - g(-\partial_\mu X^\rho \partial_\rho, \partial_\nu) - g(\partial_\mu, -\partial_\nu X^\rho \partial_\rho)$$

$$= X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\mu\rho}\partial_\nu X^\rho$$

$$\therefore (\mathcal{L}_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu X^\rho + g_{\rho\nu}\partial_\mu X^\rho$$

$\mathcal{L}_X \omega = i_X(d\omega) + d(i_X\omega)$ 을 증명하자

앞선 문제 결과를 이용해 $[i_X(d\omega) + d(i_X\omega)]_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$ 을 증명하겠다.

$$d\omega = (\partial_\alpha \omega_\beta)\, dx^\alpha \wedge dx^\beta$$

$$i_X(d\omega) = (\partial_\alpha\omega_\beta)(i_X dx^\alpha)\wedge dx^\beta + (\partial_\alpha\omega_\beta)\, dx^\alpha \wedge (i_X dx^\beta)$$

$$= (X^\alpha \partial_\alpha \omega_\beta)\, dx^\beta - (X^\beta \partial_\alpha \omega_\beta)\, dx^\alpha$$

$$= (X^\alpha \partial_\alpha \omega_\beta)\, dx^\beta - (X^\alpha \partial_\beta \omega_\alpha)\, dx^\beta$$

$$d(i_X\omega) = d(X^\alpha \omega_\alpha) = (\partial_\beta X^\alpha \omega_\alpha)\, dx^\beta = X^\alpha(\partial_\beta \omega_\alpha)\, dx^\beta + \omega_\alpha(\partial_\beta X^\alpha)\, dx^\beta$$

$$i_X(d\omega) + d(i_X\omega) = (X^\alpha \partial_\alpha \omega_\beta)\, dx^\beta - (X^\alpha \partial_\beta \omega_\alpha)\, dx^\beta + X^\alpha(\partial_\beta \omega_\alpha)\, dx^\beta + \omega_\alpha(\partial_\beta X^\alpha)\, dx^\beta$$

$$= (X^\alpha \partial_\alpha \omega_\beta)\, dx^\beta + \omega_\alpha(\partial_\beta X^\alpha)\, dx^\beta$$

$$= X^\nu \partial_\nu \omega_\mu\, dx^\mu + \omega_\nu \partial_\mu X^\nu\, dx^\mu$$

$$[i_X(d\omega) + d(i_X\omega)]_\mu = X^\nu \partial_\nu \omega_\mu + \omega_\nu \partial_\mu X^\nu$$

6. Let ω be a p-form and η a q-form. Show that the exterior derivative satisfies the properties

- $d(d\omega) = 0$
- $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta$
- $d(\varphi^* \omega) = \varphi^*(d\omega)$ where $\varphi : M \to N$ for some manifolds M and N

p-form ω 에 대해 $d^2\omega = 0$ 을 증명한다.

ω 를 명시적으로 쓰면

$$\omega = \frac{1}{p!} \sum_{i_1 \cdots i_p} \omega_{i_1, \cdots, i_p}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

증명에 앞서, $ddx^i = 0$ 을 먼저 증명.

$$d(dx^i) = \sum_j (\partial_j 1)(dx^j \wedge dx^i) = 0$$

그리고 p-form basis 의 exterior basis 를 계산해 보자.

$$d(dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p}) = \underbrace{(d^2 x^{i_1})}_{=0} \wedge (dx^{i_2} \wedge \cdots \wedge dx^{i_p}) + (-1)^1 dx^{i_1} \wedge d(x^{i_2} \wedge \cdots \wedge dx^{i_p})$$

$$= -dx^{i_1} \wedge \left\{ \underbrace{d^2 x^{i_2}}_{=0} \wedge (dx^{i_3} \wedge \cdots \wedge dx^{i_p}) - dx^{i_2} \wedge d(dx^{i_3} \wedge \cdots \wedge dx^{i_p}) \right\}$$

$$= dx^{i_1} \wedge dx^{i_2} \wedge \left\{ \underbrace{d^2 x^{i_3}}_{=0} \wedge (dx^{i_4} \wedge \cdots \wedge dx^{i_p}) - dx^{i_3} \wedge d(dx^{i_4} \wedge \cdots \wedge dx^{i_p}) \right\}$$

이 과정을 반복하면 결국

$$= (-1)^{p-1}\, dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge \underbrace{d^2 x^{i_p}}_{=0}$$

$$= 0$$

결국 모든 p-form basis 에 d를 취하면 0 이 나온다는 결론

$$d\omega = \frac{1}{p!} \sum_{i_1 \cdots i_p} (d\omega_{i_1, \cdots, i_p}(x)) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_p}) + \omega_{i_1, \cdots, i_p}(x) \underbrace{d(dx^{i_1} \wedge \cdots \wedge dx^{i_p})}_{=0}$$

$$= \frac{1}{p!} \sum_k \sum_{i_1 \cdots i_p} (\partial_k \omega_{i_1, \cdots, i_p}) dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

$$d(d\omega) = \frac{1}{p!} \sum_k \sum_{i_1, \cdots, i_p} d(\partial_k \omega_{i_1, \cdots, i_p}) \wedge (dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}) + \frac{d\omega_{i_1, \cdots, i_p}}{dx^k} \underbrace{d(dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p})}_{=0}$$

$$= \frac{1}{p!} \sum_{j,k} \sum_{i_1, \cdots, i_p} \overbrace{\underbrace{(\partial_j \partial_k \omega_{i_1, \cdots, i_p})}_{= \partial_k \partial_j \omega_{i_1, \cdots, i_p}}}^{\text{symmetric term}} \overbrace{\underbrace{(dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p})}_{= -dx^k \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}}}^{\text{anti symmetric term}}$$

$$= \frac{1}{p!} \frac{1}{2} \sum_{j,k} \sum_{i_1, \cdots, i_p} \left[(\partial_j \partial_k \omega_{i_1, \cdots, i_p})(dx^j \wedge dx^k \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}) - (\partial_k \partial_j \omega_{i_1, \cdots, i_p})(dx^k \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \right]$$

$$= 0$$

이어서 →

$d(\omega\wedge\eta)=(d\omega)\wedge\eta+(-1)^p\omega\wedge d\eta$ 을 증명하자

$\omega\wedge\eta$ 를 명시적으로 나타내는 걸 먼저.

$$\omega=\frac{1}{p!}\sum_{i_1,\cdots,i_p}\omega_{i_1,\cdots,i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$\eta=\frac{1}{q!}\sum_{j_1,\cdots,j_q}\eta_{j_1,\cdots,j_q}dx^{j_1}\wedge\cdots\wedge dx^{j_q}$$

$$\omega\wedge\eta=\frac{1}{p!q!}\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}\omega_{i_1,\cdots,i_p}\eta_{j_1,\cdots,j_q}dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}$$

$$\begin{aligned}
d(\omega\wedge\eta)=\frac{1}{p!q!}\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}&d(\omega_{i_1,\cdots,i_p}\eta_{j_1,\cdots,j_q})\wedge(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q})\\
&+(\omega_{i_1,\cdots,i_p}\eta_{j_1,\cdots,j_q})\underbrace{d(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q})}_{=0}\\
=\frac{1}{p!q!}&\sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}(\partial_k\omega_{i_1,\cdots,i_p}\eta_{j_1,\cdots,j_q})dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}\\
=\frac{1}{p!q!}&\sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}(\eta_{j_1,\cdots,j_q}\partial_k\omega_{i_1,\cdots,i_p})(dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(dx^{j_1}\wedge\cdots\wedge dx^{j_q})\\
&+\frac{1}{p!q!}\sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}(\omega_{i_1,\cdots,i_p}\partial_k\eta_{j_1,\cdots,j_q})(-1)^p(dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge(dx^k\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q})\\
=\frac{1}{p!q!}&\sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}\{(\partial_k\omega_{i_1,\cdots,i_p})dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\}\wedge(\eta_{j_1,\cdots,j_q}dx^{j_1}\wedge\cdots\wedge dx^{j_q})\\
&+\frac{1}{p!q!}\sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_q}(-1)^p(\omega_{i_1,\cdots,i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p})\wedge\{(\partial_k\eta_{j_1,\cdots,j_q})dx^k\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_q}\}\\
=(d\omega)&\wedge\eta+(-1)^p\omega\wedge d\eta
\end{aligned}$$

이어서 →

$d(\varphi^*\omega) = \varphi^*(d\omega)$ 을 증명하자.

$$\varphi^* : \Omega^p(U_2) \longrightarrow \Omega^p(U_1)$$

$$\varphi : U_1 \to U_2$$
$$x \mapsto \varphi(x) \quad , \quad \varphi^i(x) = y^i$$

U_2 에서 y로 표현되는 coordinate 를 사용, U_1 에서 x로 표현되는 coordinate 사용.

φ^* 의 변환규칙 : U_2에서 정의된 0-form f 에 대해, $\varphi^*(f(y)) = f(\varphi(x))$

U_2의 1-form basis dy^i 에 대해, $\varphi^*(dy^i) = d\varphi^*(y^i) = d\varphi^i(x) = \sum_j (\partial_j \varphi^i) dx^j$

$$\varphi^*(\omega) = \varphi^*\left(\frac{1}{p!}\sum_{j_1,\cdots,j_p} \omega_{j_1,\cdots,j_p}(y)\, dy^{j_1}\wedge\cdots\wedge dy^{j_p}\right)$$

$$= \sum_{j_1,\cdots,j_p} \varphi^*(\omega_{j_1,\cdots,j_p}(y))\ \varphi^*(dy^{j_1})\wedge\cdots\wedge\varphi^*(dy^{j_p}))$$

$$= \frac{1}{p!}\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p} \omega_{j_1,\cdots,j_p}(\varphi(x))(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})\, dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$\partial_i = \frac{\partial}{\partial x^i}$

$$d\varphi^*(\omega) = \sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p}\left\{\partial_k\, \underbrace{\omega_{j_1,\cdots,j_p}(\varphi(x))(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})}_{\frac{\partial}{\partial x^k}\text{ 미분 범위}}\right\} dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$\partial_k = \frac{\partial}{\partial x^k}$

$$= \sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p}\left\{\partial_k\,\omega_{j_1,\cdots,j_p}(\varphi(x))\right\}(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})\, dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$\left(+\sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p}\omega_{j_1,\cdots,j_p}(\varphi(x))\left\{\partial_k(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})\right\}\underline{dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}}\right)$$

$= (\partial_k\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})$
$+ (\partial_{i_1}\varphi^{j_1})(\partial_k\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})$
$\vdots$
$+ (\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_k\partial_{i_p}\varphi^{j_p})$

k와 i의 위치를 swap해도 symmetric.

k와 i의 위치를 swap 하면 antisymmetric

summation 하면 0 이 된다.

$$d\varphi^*(\omega) = \sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p}\left\{\partial_k\,\omega_{j_1,\cdots,j_p}(\varphi(x))\right\}(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})\, dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$d\omega = \frac{1}{p!}\sum_l\sum_{j_1,\cdots,j_p}(\partial_l\,\omega_{j_1,\cdots,j_p}(y))\, dy^l\wedge dy^{j_1}\wedge\cdots\wedge dy^{j_p}$$

$\partial_l = \frac{\partial}{\partial y^l}$

$$\varphi^*(d\omega) = \frac{1}{p!}\sum_l\sum_{j_1,\cdots,j_p}\varphi^*(\partial_l\,\omega_{j_1,\cdots,j_p}(y))\,\varphi^*(dy^l)\wedge\varphi^*(dy^{j_1})\wedge\cdots\wedge\varphi^*(dy^{j_p})$$

$$= \frac{1}{p!}\sum_k\sum_l\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p}\cancel{\frac{\partial x^k}{\partial\varphi^l}}\left(\partial_k\,\omega_{j_1,\cdots,j_p}(\varphi(x))\right)\cancel{\frac{d\varphi^l}{dx^k}}(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})\, dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$= \sum_k\sum_{i_1,\cdots,i_p}\sum_{j_1,\cdots,j_p}\left\{\partial_k\,\omega_{j_1,\cdots,j_p}(\varphi(x))\right\}(\partial_{i_1}\varphi^{j_1})(\partial_{i_2}\varphi^{j_2})\cdots(\partial_{i_p}\varphi^{j_p})\, dx^k\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

$$= d(\varphi^*(\omega))$$

7. The exterior derivative of $\omega \in \Lambda^p(M)$ can be defined as

$$
\begin{aligned}
d\omega(X_1,\ldots,X_{p+1}) &= \sum_{j=1}^{p+1}(-1)^{j-1}X_j(\omega(X_1,\ldots,X_{j-1},X_{j+1},\ldots,X_{p+1})) \\
&\quad + \sum_{j<k}(-1)^{j+k}\omega([X_j,X_k],X_1,\ldots,X_{j-1},X_{j+1},\ldots,X_{k-1},X_{k+1},\ldots X_{p+1})
\end{aligned}
$$

In a coordinate basis $\omega = \frac{1}{p!}\omega_{\mu_1\ldots\mu_p}dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$, use this definition to determine the components of $d\omega$ in the cases $p=1$ and $p=2$.

X 들은 임의의 벡터장 $X_n = \sum_i X_n^i \partial_i$,

$[X_n, X_m] = \sum_i\sum_j X_n^i \partial_i X_m^j \partial_j - X_m^j \partial_j X_n^i \partial_i = \sum_i\sum_j \{(\partial_i X_m^j)X_n^i - (\partial_i X_n^j)X_m^i\}\partial_j$

$[X_n, X_m]^\mu = \sum_i \{(\partial_i X_m^\mu)X_n^i - (\partial_i X_n^\mu)X_m^i\}$

in $p=1$ case,

$\omega = \sum_{\mu_1}\omega_{\mu_1}dx^{\mu_1}$, $\omega(X_n) = \sum_{\mu_1}\omega_{\mu_1}X_n^{\mu_1}$, $X_m(\omega(X_n)) = \sum_{\mu_1}\sum_i X_m^i\partial_i(\omega_{\mu_1}X_n^{\mu_1})$

$d\omega = \sum_{\mu_1,\mu_2}(\partial_{\mu_2}\omega_{\mu_1})dx^{\mu_2}\wedge dx^{\mu_1}$ 임을 증명하겠다.

$$d\omega(X_1,X_2) = \sum_{\mu_1,\mu_2}(X_1^{\mu_2}X_2^{\mu_1} - X_2^{\mu_2}X_1^{\mu_1})(\partial_{\mu_2}\omega_{\mu_1})$$

문제의 정의에 따라,

$$
\begin{aligned}
d\omega(X_1,X_2) &= X_1(\omega(X_2)) - X_2(\omega(X_1)) - \omega([X_1,X_2]) \\
&= \sum_{\mu_1}\sum_i X_1^i\partial_i(\omega_{\mu_1}X_2^{\mu_1}) - \sum_{\mu_1}\sum_i X_2^i\partial_i(\omega_{\mu_1}X_1^{\mu_1}) - \sum_{\mu_1}\sum_i \omega_{\mu_1}\{(\partial_i X_2^{\mu_1})X_1^i - (\partial_i X_1^{\mu_1})X_2^i\} \\
&= \sum_{\mu_1}\sum_i\left[\cancel{X_1^i\omega_{\mu_1}(\partial_i X_2^{\mu_1})} + X_1^iX_2^{\mu_1}(\partial_i\omega_{\mu_1}) - \cancel{X_2^i\omega_{\mu_1}(\partial_i X_1^{\mu_1})} - X_2^iX_1^{\mu_1}(\partial_i\omega_{\mu_1}) - \cancel{X_1^i\omega_{\mu_1}(\partial_i X_2^{\mu_1})} + \cancel{X_2^i\omega_{\mu_1}(\partial_i X_1^{\mu_1})}\right] \\
&= \sum_{\mu_1}\sum_i\left[X_1^iX_2^{\mu_1}(\partial_i\omega_{\mu_1}) - X_2^iX_1^{\mu_1}(\partial_i\omega_{\mu_1})\right] \\
&= \sum_{\mu_1,\mu_2}(X_1^{\mu_2}X_2^{\mu_1} - X_2^{\mu_2}X_1^{\mu_1})(\partial_{\mu_2}\omega_{\mu_1})
\end{aligned}
$$

이어서

in $p=2$ case

$$\omega = \sum_{\mu_1,\mu_2} \frac{1}{2}\, \omega_{\mu_1\mu_2}\, dx^{\mu_1} \wedge dx^{\mu_2}$$

$$d\omega = \frac{1}{2}\sum_{\mu_1\mu_2\mu_3} \left(\partial_{\mu_3}\omega_{\mu_1\mu_2}\right) dx^{\mu_3}\wedge dx^{\mu_1}\wedge dx^{\mu_2} = \frac{1}{2}\sum_{\mu_1\mu_2\mu_3} \left(\partial_{\mu_3}\omega_{\mu_1\mu_2}\right) dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\mu_3}$$

위 $d\omega$ 의 명시적표현이 옳음을 증명하겠다.

문제에서 주어진 정의에 따르면,

$$d\omega(X_1,X_2,X_3) = X_1(\omega(X_2,X_3)) - X_2(\omega(X_1,X_3)) + X_3(\omega(X_1,X_2))$$
$$- \omega([X_1,X_2],X_3) + \omega([X_1,X_3],X_2) - \omega([X_2,X_3],X_1)$$

각 항을 전개,

$$X_1(\omega(X_2,X_3)) = \sum_i \sum_{\mu,\nu} X_1^i\, \partial_i\left(\omega_{\mu\nu} X_2^\mu X_3^\nu\right)$$
$$= \sum_i\sum_{\mu,\nu} X_1^i \left[X_2^\mu X_3^\nu (\partial_i \omega_{\mu\nu}) + \omega_{\mu\nu} X_3^\nu (\partial_i X_2^\mu) + \omega_{\mu\nu} X_2^\mu (\partial_i X_3^\nu)\right]$$

$$-X_2(\omega(X_1,X_3)) = \sum_{i,\mu,\nu} X_2^i \left[- X_1^\mu X_3^\nu (\partial_i \omega_{\mu\nu}) - \omega_{\mu\nu} X_3^\nu (\partial_i X_1^\mu) - \omega_{\mu\nu} X_1^\mu (\partial_i X_3^\nu)\right]$$

$$X_3(\omega(X_1,X_2)) = \sum_{i,\mu,\nu} X_3^i \left[X_1^\mu X_2^\nu (\partial_i \omega_{\mu\nu}) + \omega_{\mu\nu} X_2^\nu (\partial_i X_1^\mu) + \omega_{\mu\nu} X_1^\mu (\partial_i X_2^\nu)\right]$$

$$-\omega([X_1,X_2],X_3) = \sum_{\mu\nu} \omega_{\mu\nu} [X_1,X_2]^\mu X_3^\nu = \sum_{i\mu\nu} \omega_{\mu\nu} X_3^\nu \left[-X_1^i(\partial_i X_2^\mu) + X_2^i(\partial_i X_1^\mu)\right]$$

$$\omega([X_1,X_3],X_2) = \sum_{i\mu\nu} \omega_{\mu\nu} X_2^\nu \left[X_1^i(\partial_i X_3^\mu) - X_3^i(\partial_i X_1^\mu)\right]$$

$$\omega_{\mu\nu} X_2^\nu X_1^i (\partial_i X_3^\mu) = -\omega_{\mu\nu} X_2^\mu X_1^i (\partial_i X_3^\nu)$$

$$-\omega([X_2,X_3],X_1) = \sum_{i\mu\nu} \omega_{\mu\nu} X_1^\nu \left[-X_2^i(\partial_i X_3^\mu) + X_3^i(\partial_i X_2^\mu)\right]$$

$$\omega_{\mu\nu} X_1^\nu X_3^i (\partial_i X_2^\mu) = \omega_{\mu\nu} X_1^\mu X_3^i (\partial_i X_2^\nu)$$

$$-\omega_{\mu\nu} X_1^\nu X_2^i (\partial_i X_3^\mu) = -\omega_{\nu\mu} X_1^\mu X_2^i (\partial_i X_3^\nu) = \omega_{\mu\nu} X_1^\mu X_2^i (\partial_i X_3^\nu)$$

$d\omega(X_1,X_2,X_3)$ 에서 같은 색의 항들 끼리 상쇄되어 사라 진다. ($\omega_{\mu\nu} = -\omega_{\nu\mu}$ 를 이용)

결국 남는 것은,

$$d\omega(X_1,X_2,X_3) = X_1^i X_2^\mu X_3^\nu (\partial_i\omega_{\mu\nu}) - X_2^i X_1^\mu X_3^\nu (\partial_i \omega_{\mu\nu}) + X_3^i X_1^\mu X_2^\nu (\partial_i \omega_{\mu\nu})$$
$$= X_1^i X_2^\mu X_3^\nu (\partial_i\omega_{\mu\nu}) + X_2^i X_3^\mu X_1^\nu (\partial_i \omega_{\mu\nu}) + X_3^i X_1^\mu X_2^\nu (\partial_i \omega_{\mu\nu})$$
$$= \frac{1}{2}\sum_{\mu_1\mu_2\mu_3} \left(\partial_{\mu_3}\omega_{\mu_1\mu_2}\right) dx^{\mu_1}\wedge dx^{\mu_2}\wedge dx^{\mu_3}$$

증명끝.

8. A three-sphere can be parameterized by Euler angles (θ, ϕ, ψ) where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. Define the following 1-forms

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi\ , \quad \sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi\ , \quad \sigma_3 = d\psi + \cos\theta\, d\phi$$

Show that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ with analogous results for $d\sigma_2$ and $d\sigma_3$.

$$d\sigma_1 = \left(\frac{\partial}{\partial\psi} -\sin\psi\right) d\psi \wedge d\theta + \left(\frac{\partial}{\partial\psi} \cos\psi \sin\theta\right) d\psi \wedge d\phi + \left(\frac{\partial}{\partial\theta} \cos\psi \sin\theta\right) d\theta \wedge d\phi$$

$$= -\cos\psi\, d\psi \wedge d\theta + \sin\psi \sin\theta\, d\phi \wedge d\psi + \cos\psi \cos\theta\, d\theta \wedge d\phi$$

$$d\sigma_2 = \left(\frac{\partial}{\partial\psi} \cos\psi\right) d\psi \wedge d\theta + \left(\frac{\partial}{\partial\psi} \sin\psi \sin\theta\right) d\psi \wedge d\phi + \left(\frac{\partial}{\partial\theta} \sin\psi \sin\theta\right) d\theta \wedge d\phi$$

$$= -\sin\psi\, d\psi \wedge d\theta - \cos\psi \sin\theta\, d\phi \wedge d\psi + \sin\psi \cos\theta\, d\theta \wedge d\phi$$

$$d\sigma_3 = \left(\frac{\partial}{\partial\theta} \cos\theta\right) d\theta \wedge d\phi$$

$$= -\sin\theta\, d\theta \wedge d\phi$$

$$\sigma_2 \wedge \sigma_3 = (\cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi) \wedge (d\psi + \cos\theta\, d\phi)$$

$$= -\cos\psi\, d\psi \wedge d\theta + \sin\psi \sin\theta\, d\phi \wedge d\psi + \cos\psi \cos\theta\, d\theta \wedge d\phi$$

$$= d\sigma_1$$

$$\sigma_3 \wedge \sigma_1 = (d\psi + \cos\theta\, d\phi) \wedge (-\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi)$$

$$= -\sin\psi\, d\psi \wedge d\theta + \sin\psi \cos\theta\, d\theta \wedge d\phi - \cos\psi \sin\theta\, d\phi \wedge d\psi$$

$$= d\sigma_2$$

$$\sigma_1 \wedge \sigma_2 = (-\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi) \wedge (\cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi)$$

$$= -\sin^2\psi \sin\theta\, d\theta \wedge d\phi - \cos^2\psi \sin\theta\, d\theta \wedge d\phi$$

$$= -\sin\theta\, d\theta \wedge d\phi$$

$$= d\sigma_3$$