QM 11th week HW   20262182 김신지

Sakurai 4.2.

T : $d\mathbb{l}$ − direction translation operator

$D(\hat{n}, \phi)$: rotation operation (axis : $\hat{n}$, angle : ϕ)

Π : parity operator

다음 쌍 중에서 commute 하는 쌍이 있나? 있다면 왜 그렇나?

a. $T_{d\mathbb{l}}$ and $T_{d\mathbb{l}'}$ b. $D(\hat{n}, \phi)$ and $D(\hat{n}', \phi')$

c. $T_{d\mathbb{l}}$ and Π d. $D(\hat{n}, \phi)$ and Π

→ 풀이

a. 서로 다른 방향의 translation operator는 commute 한다.

x, y, z 축의 momentum operator가 서로 commute 하기 때문이다.

$$d\mathbb{l} = d_x\hat{x} + d_y\hat{y} + d_z\hat{z} \qquad d\mathbb{l}' = d_x'\hat{x} + d_y'\hat{y} + d_z'\hat{z}$$

$$\mathbb{P}\cdot d\mathbb{l} = d_xP_x + d_yP_y + d_zP_z$$

$d\mathbb{l}$와 $d\mathbb{l}'$가 infinitesimal 하다면,

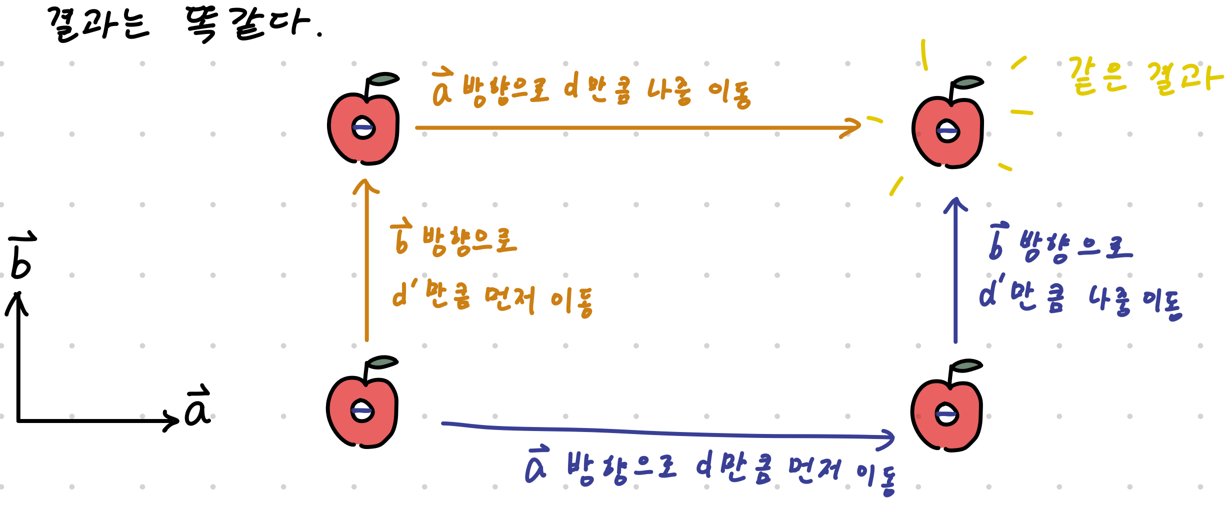
$$T_{d\mathbb{l}} = 1 - \frac{i}{\hbar}\mathbb{P}\cdot d\mathbb{l} = 1 - \frac{i}{\hbar}(d_xP_x + d_yP_y + d_zP_z)$$

$$[T_d, T_{d'}] = \left[1 - \frac{i}{\hbar}(d_xP_x + d_yP_y + d_zP_z),\ 1 - \frac{i}{\hbar}(d_x'P_x + d_y'P_y + d_z'P_z)\right]$$

$$= -\frac{1}{\hbar^2}\left[d_xP_x + d_yP_y + d_zP_z,\ d_x'P_x + d_y'P_y + d_z'P_z\right]$$

$[P_i, P_j] = 0$ for $\forall i, j = x, y, z$ 이므로, 위 식은 0. $\therefore [T_d, T_{d'}] = 0$

고전적으로도 서로 직교하는 두 방향 $\vec{a}$ 와 $\vec{b}$에 대해, 어느쪽 방향으로 translation을 진행하든 결과는 똑같다.



b. $D(\hat{n},\phi)$ and $D(\hat{n}',\phi')$

Momentum과 달리 angular momentum은 서로 다른 방향의 operator가 commut 하지 않기 때문이다. $[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$.

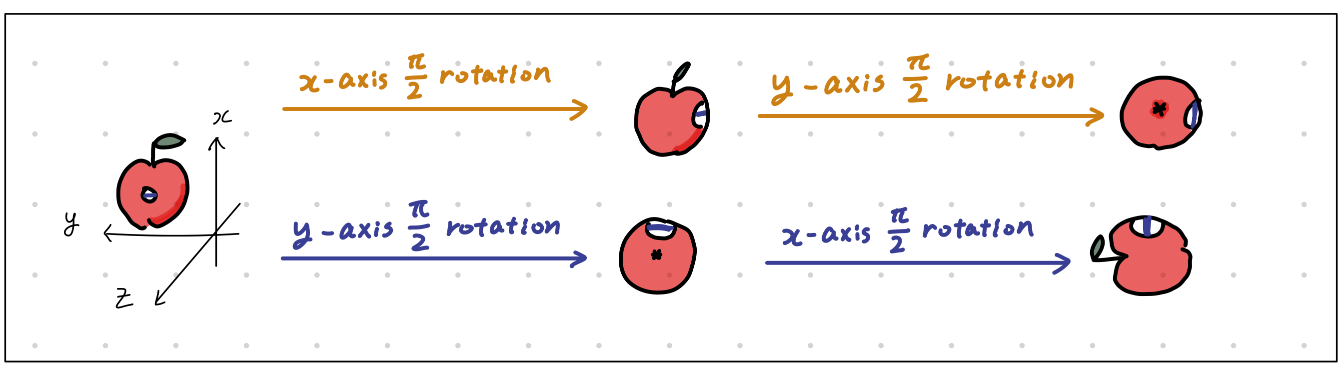
문제에서 $\hat{n} = \hat{x}$, $\hat{n} = \hat{y}$, $\phi = \phi'$ 로 설정한 예시를 강의 중에 다뤘다.

$$[D(x,\phi), D(y,\phi)] = \left(1 - i\frac{\phi}{\hbar}J_x - \frac{\phi^2}{2\hbar^2}J_x^2\right)\left(1 - i\frac{\phi}{\hbar}J_y - \frac{\phi^2}{2\hbar^2}J_y^2\right) - \left(1 - i\frac{\phi}{\hbar}J_y - \frac{\phi^2}{2\hbar^2}J_y^2\right)\left(1 - i\frac{\phi}{\hbar}J_x - \frac{\phi^2}{2\hbar^2}J_x^2\right)$$

$$= 1 - \frac{\phi^2}{\hbar^2}[J_x, J_y] - 1 = -i\frac{\phi^2}{\hbar}J_z \simeq D(z,\phi^2) - 1$$

일반적으로도 $\hat{n}$과 $\hat{n}'$이 평행하지 않은 경우 $(|\hat{n}\cdot\hat{n}'| \neq 1)$, $D(\hat{n},\phi)$과 $D(\hat{n}',\phi')$은 commute 하지 않는다.

고전적으로도, 일반적으로 회전 과정의 순서를 바꾸면 결과가 달라진다.



C. T_d and Π

translation과 parity는 commute 하지 않는다.

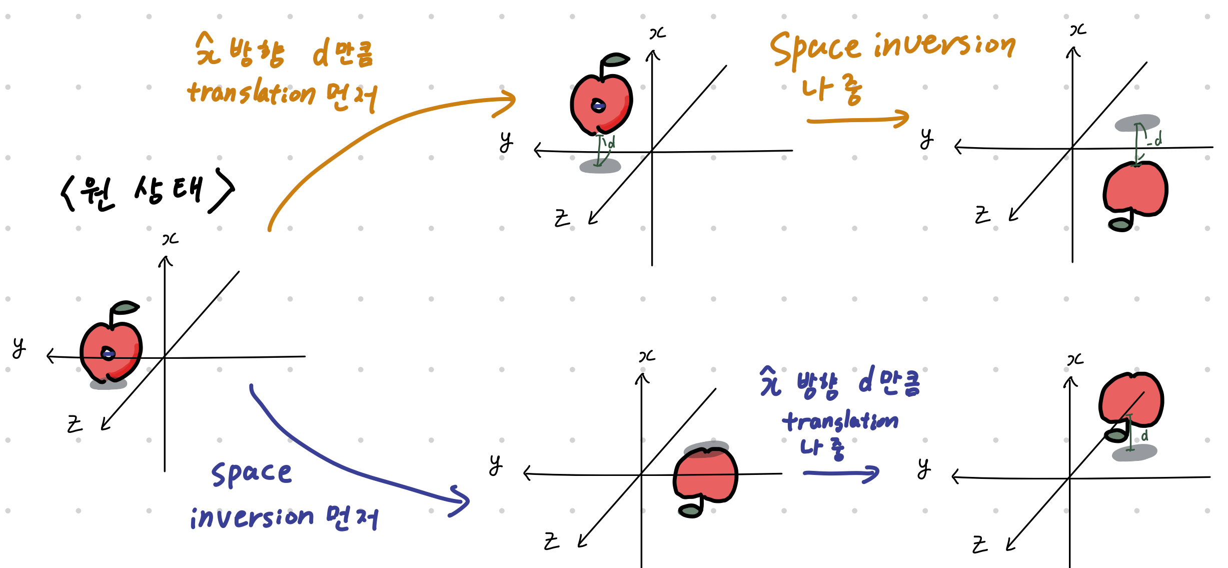
$T_d = 1 - \frac{i}{\hbar}P\cdot d$ 에서, d는 operator가 아니므로, $d\Pi = \Pi d$

momentum은 parity에 odd이므로, $\Pi P \Pi^\dagger = -P$, $\Pi P = -P\Pi$

$$\Pi T_d \Pi^\dagger = \Pi\left(1 - \frac{i}{\hbar}P\cdot d\right)\Pi^\dagger = 1 + \frac{i}{\hbar}P\cdot d = T_{-d}$$

$\therefore \Pi T_d = T_{-d}\Pi$

고전적으로도, translation을 먼저하고 space inversion을 한 것과, 그 반대 순서로 한 것은 결과가 다르다.



→ 이건 space inversion 이후 x̂ 방향 -d translation 과 같다.

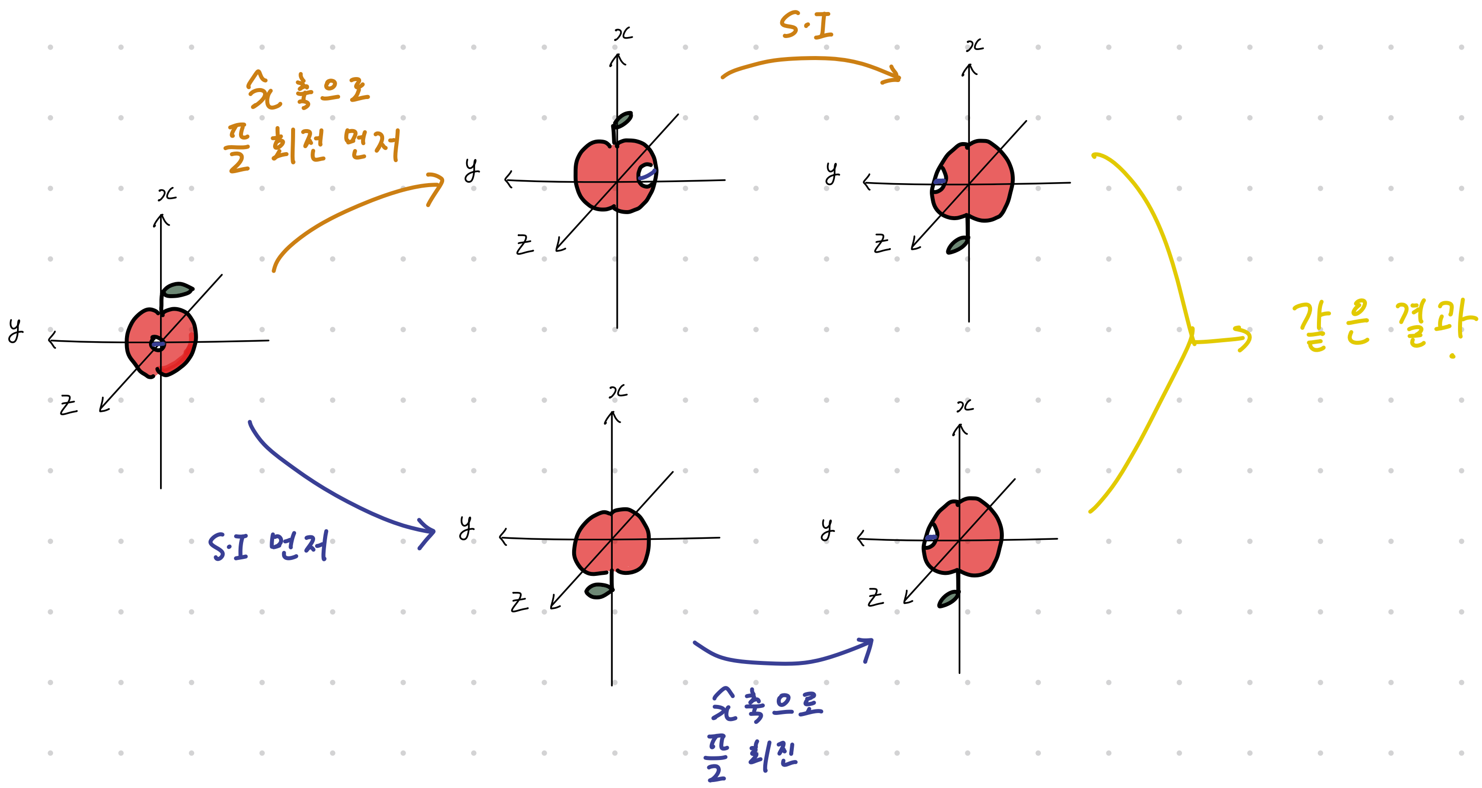
d. $D(\hat{n}, \phi)$ and Π

Momentum과 달리 angular momentum은 parity에 even 하다.

$\Pi \vec{J} \Pi^{\dagger} = \vec{J}$ 따라서 rotation operator와 parity operator는 commute 하다.

$$\Pi\left(1 - \frac{i}{\hbar}\vec{J}\cdot\hat{n}\right)\Pi^{\dagger} = 1 - \frac{i}{\hbar}\vec{J}\cdot\hat{n} \qquad \Pi D(\hat{n},\phi) = D(\hat{n},\phi)\,\Pi$$

고전적으로 확인해도 그렇다.



Sakurai 4.7

a. $\psi(\mathbf{x},t)$가 3-D에서 Spinless particle의 plane wave에 해당하는 wavefunction.
$\psi^{*}(\mathbf{x},-t)$는 momentum 방향이 반전되었음을 보여라.

풀이 → $\psi(\mathbf{x},t) = \langle \mathbf{x}|\alpha,t\rangle$, $\psi^{*}(\mathbf{x},-t) = \langle \mathbf{x}|\Theta|\alpha,t\rangle$, $\phi(\mathbf{p},t) = \langle \mathbf{p}|\alpha,t\rangle$, $\tilde{\phi}(\mathbf{p},-t) = \langle \mathbf{p}|\Theta|\alpha,t\rangle$

ψ와 ψ^{*}에 대한 푸리에 변환을 통해 $\phi^{*}(-p) = \tilde{\phi}(p)$ 임을 보이겠다.

$$\phi(p,t) = \int dx\, e^{-ipx}\psi(x,t)$$

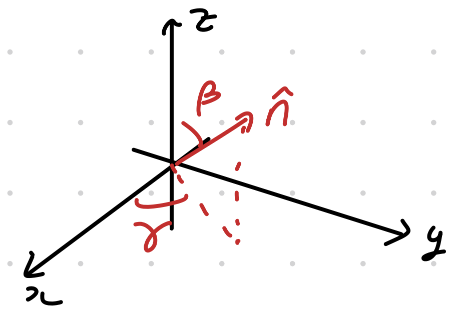
$$\tilde{\phi}(p,-t) = \int dx\, e^{-ipx}\psi^{*}(x,t) ,$$

$$\tilde{\phi}^{*}(p,-t) = \int dx\, e^{ipx}\psi(x,-t) = \int dx\, e^{-i(-p)x}\psi(x,-t) = \phi(-p,-t)$$

$$\therefore \tilde{\phi}(p,-t) = \phi^{*}(-p,-t)$$

b. $\chi(\hat{n})$은 $\vec{\sigma}\cdot\hat{n}$ 의 two-component eigen spinnor. Eigen value 는 1.

polar angle β 와 azimuthal angle γ 를 이용해 $\chi(\hat{n})$의 explicit form을 나타내고, $-i\sigma_y\chi^*(\hat{n})$ 가 방향이 반대인 two-component eigen spinnor 임을 보여라.



$$\hat{n} = (\sin\beta\cos\gamma, \sin\beta\sin\gamma, \cos\beta)$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D(\hat{n},\phi) = \mathbb{1}\cos\frac{\phi}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\phi}{2}$$

$$\chi(\hat{n}) = D(z,\gamma)D(y,\beta)\chi_+ = \begin{pmatrix} \cos\frac{\gamma}{2} - i\sin\frac{\gamma}{2} & 0 \\ 0 & \cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \exp(-i\frac{\gamma}{2}) & 0 \\ 0 & \exp(i\frac{\gamma}{2}) \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} \\ \sin\frac{\beta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \exp(-i\frac{\gamma}{2})\cos\frac{\beta}{2} \\ \exp(i\frac{\gamma}{2})\sin\frac{\beta}{2} \end{pmatrix}$$

$$\tilde{\chi}(\hat{n}) = -i\sigma_y\chi^*(\hat{n}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \exp(-i\frac{\gamma}{2})\cos\frac{\beta}{2} \\ \exp(i\frac{\gamma}{2})\sin\frac{\beta}{2} \end{pmatrix} = \begin{pmatrix} -\exp(i\frac{\gamma}{2})\sin\frac{\beta}{2} \\ \exp(-i\frac{\gamma}{2})\cos\frac{\beta}{2} \end{pmatrix}$$

일단 $\tilde{\chi}(\hat{n})$ 가 $\begin{pmatrix}0\\1\end{pmatrix}$ spinnor 를 y축에 대해 β 돌리고, z 축에 대해 γ 돌린 결과와 같음을 보이겠다.

$$\tilde{\chi}(\hat{n}) = D(z,\gamma)D(y,\beta)\chi_- = \begin{pmatrix} \exp(-i\frac{\gamma}{2}) & 0 \\ 0 & \exp(i\frac{\gamma}{2}) \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \exp(-i\frac{\gamma}{2}) & 0 \\ 0 & \exp(i\frac{\gamma}{2}) \end{pmatrix}\begin{pmatrix} -\sin\frac{\beta}{2} \\ \cos\frac{\beta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\exp(-i\frac{\gamma}{2})\sin\frac{\beta}{2} \\ \exp(i\frac{\gamma}{2})\cos\frac{\beta}{2} \end{pmatrix}$$

그리고 $\vec{\sigma}\cdot\hat{n}$ 의 eigen spinnor 임을 행렬 표현으로 직접 보이겠다.

$$\hat{n} = (\sin\beta\cos\gamma, \sin\beta\sin\gamma, \cos\beta)$$

$$\vec{\sigma}\cdot\hat{n} = \begin{pmatrix} \cos\beta & \sin\beta\cos\gamma - i\sin\beta\sin\gamma \\ \sin\beta\cos\gamma + i\sin\beta\sin\gamma & -\cos\beta \end{pmatrix}$$

$$(\vec{\sigma}\cdot\hat{n})\tilde{\chi}(\hat{n}) = \begin{pmatrix} \cos\beta & \sin\beta\cos\gamma - i\sin\beta\sin\gamma \\ \sin\beta\cos\gamma + i\sin\beta\sin\gamma & -\cos\beta \end{pmatrix}\begin{pmatrix} (-\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2})\sin\frac{\beta}{2} \\ (\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2})\cos\frac{\beta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta(-\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2})\sin\frac{\beta}{2} + (\sin\beta\cos\gamma - i\sin\beta\sin\gamma)(\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2})\cos\frac{\beta}{2} \\ (\sin\beta\cos\gamma + i\sin\beta\sin\gamma)(-\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2})\sin\frac{\beta}{2} - \cos\beta(\cos\frac{\gamma}{2} + i\sin\frac{\gamma}{2})\cos\frac{\beta}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -\cos\beta\cos\frac{\gamma}{2}\sin\frac{\beta}{2} + \sin\beta\cos\gamma\cos\frac{\gamma}{2}\cos\frac{\beta}{2} + \sin\beta\sin\gamma\sin\frac{\gamma}{2}\cos\frac{\beta}{2} \\ + i\left(\cos\beta\sin\frac{\gamma}{2}\sin\frac{\beta}{2} - \sin\beta\sin\gamma\cos\frac{\gamma}{2}\cos\frac{\beta}{2} + \sin\beta\cos\gamma\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\right) \\ \\ -\sin\beta\cos\gamma\cos\frac{\gamma}{2}\sin\frac{\beta}{2} - \sin\beta\sin\gamma\sin\frac{\gamma}{2}\sin\frac{\beta}{2} - \cos\beta\cos\frac{\gamma}{2}\cos\frac{\beta}{2} \\ + i\left(-\sin\beta\sin\gamma\cos\frac{\gamma}{2}\sin\frac{\beta}{2} + \sin\beta\cos\gamma\sin\frac{\gamma}{2}\sin\frac{\beta}{2} - \cos\beta\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\right. \end{pmatrix}$$

첫번째 요소

$$-\cos\beta\cos\frac{\gamma}{2}\sin\frac{\beta}{2}+\underbrace{\sin\beta\cos\gamma\cos\frac{\gamma}{2}\cos\frac{\beta}{2}+\sin\beta\sin\gamma\sin\frac{\gamma}{2}\cos\frac{\beta}{2}}_{\sin\beta\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$$

$$+i\left(\cos\beta\sin\frac{\gamma}{2}\sin\frac{\beta}{2}\underbrace{-\sin\beta\sin\gamma\cos\frac{\gamma}{2}\cos\frac{\beta}{2}+\sin\beta\cos\gamma\sin\frac{\gamma}{2}\cos\frac{\beta}{2}}_{-\sin\beta\cos\frac{\beta}{2}\sin\frac{\gamma}{2}}\right)$$

$$=\cos\frac{\gamma}{2}\sin\frac{\beta}{2}-i\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$=\left(\cos\frac{\gamma}{2}-i\sin\frac{\gamma}{2}\right)\sin\frac{\beta}{2}$$

두번째 요소

$$\underbrace{-\sin\beta\cos\gamma\cos\frac{\gamma}{2}\sin\frac{\beta}{2}-\sin\beta\sin\gamma\sin\frac{\gamma}{2}\sin\frac{\beta}{2}}_{-\sin\beta\sin\frac{\beta}{2}\cos\frac{\gamma}{2}}-\cos\beta\cos\frac{\gamma}{2}\cos\frac{\beta}{2}$$

$$+i\left(\underbrace{-\sin\beta\sin\gamma\cos\frac{\gamma}{2}\sin\frac{\beta}{2}+\sin\beta\cos\gamma\sin\frac{\gamma}{2}\sin\frac{\beta}{2}}_{-\sin\beta\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}-\cos\beta\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\right.$$

$$=-\cos\frac{\gamma}{2}\cos\frac{\beta}{2}-i\sin\frac{\gamma}{2}\cos\frac{\beta}{2}$$

$$=-\left(\cos\frac{\gamma}{2}+i\sin\frac{\gamma}{2}\right)\cos\frac{\beta}{2}$$

$$(\vec{\sigma}\cdot\hat{n})\tilde{\chi}(\hat{n})=\begin{pmatrix}\left(\cos\frac{\gamma}{2}-i\sin\frac{\gamma}{2}\right)\sin\frac{\beta}{2}\\ -\left(\cos\frac{\gamma}{2}+i\sin\frac{\gamma}{2}\right)\cos\frac{\beta}{2}\end{pmatrix}=-\tilde{\chi}(\hat{n})$$