QM 12th week Homework. 20262182 김신지.

Sakurai 3.2

Find the eigenvalues and eigen vectors of $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Suppose an electron is in the spin state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

If S_y is measured, what is the probability of the result $\frac{\hbar}{2}$?

풀이

Eigen value λ 찾기

$$\det\begin{pmatrix} 0 & -i-\lambda \\ i-\lambda & 0 \end{pmatrix} = (i-\lambda)(i+\lambda) = -1-\lambda^2 = 0$$

λ가 실수라면, eigen value 는 1 과 -1.

Eigen vector 찾기

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} = i\begin{pmatrix} -b_\pm \\ a_\pm \end{pmatrix} = \pm\begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix}$$

$i\begin{pmatrix} -b_+ \\ a_+ \end{pmatrix} = \begin{pmatrix} a_+ \\ b_+ \end{pmatrix}$, $|a_+|^2 + |b_+|^2 = 1$

$a_+ = \frac{1}{\sqrt{2}}$, $b_+ = \frac{i}{\sqrt{2}}$ 이면 조건을 만족.

$i\begin{pmatrix} -b_- \\ a_- \end{pmatrix} = \begin{pmatrix} -a_- \\ -b_- \end{pmatrix}$, $|a_-|^2 + |b_-|^2 = 1$

$a_- = \frac{1}{\sqrt{2}}$, $b_- = \frac{-i}{\sqrt{2}}$ 이면 조건을 만족.

$$\sigma_y\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}, \quad \sigma_y\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} = -\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -i\frac{1}{\sqrt{2}} \end{pmatrix}$$

angular momentum operator in y-axis

$$S_y = \frac{\hbar}{2}\sigma_y$$

spin state $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$가 $\frac{\hbar}{2}$로 관측될 확률은

이것이 $\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix}$ 상태에 있을 확률이다.

$$\text{Probability} = \left|\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right|^2 = \frac{1}{2}\left|\alpha - i\beta\right|^2 = \frac{1}{2}(\alpha - i\beta)(\alpha^* + i\beta^*)$$

$$= \frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + i\left(\alpha\beta^* - \alpha^*\beta\right)\right)$$

$$= \frac{1}{2} - \mathrm{Im}(\alpha\beta^*)$$

Sakurai 3.10

$$D^{(1/2)}(\alpha,\beta,\gamma) = \exp\left(-i\sigma_3 \frac{\alpha}{2}\right)\exp\left(-i\sigma_2 \frac{\beta}{2}\right)\exp\left(-i\sigma_3 \frac{\gamma}{2}\right)$$

$$= \begin{pmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{pmatrix}$$

Group property 에 의해, 이 회전은 어떤 하나의 회전 축 $\hat{n}$ 에서 각도 θ 만큼의 회전과 같을 것이다. θ를 찾아라.

풀이 →

$$D^{(1/2)}(\hat{n},\theta) = \mathbb{1}\cos\left(\frac{\theta}{2}\right) - i(\vec{\sigma}\cdot\hat{n})\sin\left(\frac{\theta}{2}\right)$$

$$D^{(1/2)}(\hat{n},\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right) & (-in_x - n_y)\sin\left(\frac{\theta}{2}\right) \\ (-in_x + n_y)\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) + in_z\sin\left(\frac{\theta}{2}\right) \end{pmatrix} = D^{(1/2)}(\alpha,\beta,\gamma)$$

$e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2}$ 와 $\cos\left(\frac{\theta}{2}\right) - in_z\sin\left(\frac{\theta}{2}\right)$ 의 실수부와 허수부를 비교하고,

$e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2}$ 와 $(-in_x + n_y)\sin\left(\frac{\theta}{2}\right)$ 의 실수부와 허수부를 비교하라.

$$\begin{cases} \cos\left(\frac{\theta}{2}\right) = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2} \\ n_z\sin\left(\frac{\theta}{2}\right) = \cos\frac{\beta}{2}\sin\frac{\alpha+\gamma}{2} \\ n_y\sin\left(\frac{\theta}{2}\right) = \sin\frac{\beta}{2}\cos\frac{\alpha-\gamma}{2} \\ n_x\sin\left(\frac{\theta}{2}\right) = -\sin\frac{\beta}{2}\sin\left(\frac{\alpha-\gamma}{2}\right) \end{cases}$$

$$\cos^2\left(\frac{\theta}{2}\right) = \cos^2\frac{\beta}{2}\cos^2\frac{\alpha+\gamma}{2}$$

$$(n_z^2+n_y^2+n_x^2)\sin^2\left(\frac{\theta}{2}\right) = \cos^2\frac{\beta}{2}\sin^2\frac{\alpha+\gamma}{2} + \sin^2\frac{\beta}{2}\cos^2\frac{\alpha-\gamma}{2} + \sin^2\frac{\beta}{2}\sin^2\frac{\alpha-\gamma}{2} = \cos^2\frac{\beta}{2}\sin^2\frac{\alpha+\gamma}{2} + \sin^2\frac{\beta}{2}$$

$$\sin^2\frac{\theta}{2} = \cos^2\frac{\beta}{2}\sin^2\frac{\alpha+\gamma}{2} + \sin^2\frac{\beta}{2}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos^2\frac{\beta}{2}\cos^2\frac{\alpha+\gamma}{2} - \cos^2\frac{\beta}{2}\sin^2\frac{\alpha+\gamma}{2} - \sin^2\frac{\beta}{2}$$

$$= \cos^2\frac{\beta}{2}\left(\cos^2\frac{\alpha+\gamma}{2} - \sin^2\frac{\alpha+\gamma}{2}\right) - \sin^2\frac{\beta}{2}$$

$$= \cos^2\frac{\beta}{2}\cos(\alpha+\gamma) - \sin^2\frac{\beta}{2}$$

$$\boxed{\therefore\ \cos\theta = \cos^2\frac{\beta}{2}\cos(\alpha+\gamma) - \sin^2\frac{\beta}{2}}$$

... 사실 이런 식의 표현이라면, $\cos\left(\frac{\theta}{2}\right) = \cos\frac{\beta}{2}\cos\frac{\alpha+\gamma}{2}$ 에서 이미 답이 나온 셈이다.

Sakurai 3.17

$|j, m=m_{max}=j\rangle$ 가 y축에 대해 ε 만큼 돌려졌다.

$d^{(j)}_{m'm}$ 함수의 명시적인 형태를 쓰지 말고,

돌려진 상태가 원래 상태로 관찰될 확률을 ε^2의 order로 나타내라.

풀이 →

y축 회전 연산의 행렬표현에서 0행 0열의 요소값을

ε^2 까지 구하는 것.

$$\exp\left(-i\frac{1}{\hbar}J_y\varepsilon\right) \simeq 1 - i\frac{1}{\hbar}J_y\varepsilon - \frac{1}{2}\frac{1}{\hbar^2}J_y^2\varepsilon^2$$

$$J_y = \frac{J_+ - J_-}{2i}$$

$$J_y^2 = -\frac{1}{4}(J_+ - J_-)(J_+ - J_-) = -\frac{1}{4}(J_+J_+ - J_-J_+ - J_+J_- + J_-J_-)$$

$$\langle j,j|J_y|j,j\rangle = \frac{1}{2i}\langle j,j|J_+ - J_-|j,j\rangle = 0$$

$$\langle j,j|J_y^2|j,j\rangle = -\frac{1}{4}\langle j,j|J_+J_+ - J_-J_+ - J_+J_- + J_-J_-|j,j\rangle$$

$$= \frac{1}{4}\langle j,j|J_+J_-|j,j\rangle$$

$$= \frac{1}{4}C^+_{j,j-1}C^-_{j,j}$$

$$= \frac{1}{4}\hbar^2(\sqrt{2j})^2$$

$$= \frac{1}{2}j\hbar^2$$

$$\boxed{\begin{aligned} C^+_{jm} &= \hbar\sqrt{(j-m)(j+m+1)} \\ C^-_{jm} &= \hbar\sqrt{(j+m)(j-m+1)} \end{aligned}}$$

$$\langle j,j|\exp\left(-i\frac{1}{\hbar}J_y\varepsilon\right)|j,j\rangle \simeq 1 - \frac{1}{2}\frac{\varepsilon^2}{\hbar^2}\langle j,j|J_y^2|j,j\rangle = 1 - \frac{j}{4}\varepsilon^2$$

$|j,j\rangle$ 상태로 관측될 확률 $= \left|\langle j,j|\exp\left(-i\frac{1}{\hbar}J_y\varepsilon\right)|j,j\rangle\right|^2 \simeq 1 - \frac{j}{2}\varepsilon^2$