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3.26 Consider an orbital angular-momentum eigenstate $|l=2, m=0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the new state to be found in $m=0, \pm 1$, and ± 2. (The spherical harmonics for $l=0,1$, and 2 given in Section B.5 in Appendix B may be useful.)

Wigner d-matrix: $d^{(j)}_{m'm}(\beta) = \langle j\, m' | D_y(\beta) | j\, m \rangle$

$|d^{(2)}_{00}(\beta)|^2$, $|d^{(2)}_{\pm 1 0}(\beta)|^2$, $|d^{(2)}_{\pm 2 0}(\beta)|^2$ 를 구하자.

d-matrix 와 spherical coordinate 사이 관계식을 유도.

$$|\hat{n} = (\phi, \theta)\rangle = D(\alpha=\phi, \beta=\theta, \gamma=0)\, |\hat{z}\rangle$$

$$\langle l, m' | \hat{n} \rangle = \sum_m \langle l\, m' | D(\alpha=\phi, \beta=\theta, \gamma=0) | l\, m\rangle \langle l\, m | \hat{z}\rangle$$
$$= \sum_m D^{l}_{m'm}(\alpha=\phi, \beta=\theta, \gamma=0) \langle l\, m | \hat{z}\rangle$$

$$\langle l\, m | \hat{z}\rangle = Y_l^{m*}(\theta=0, \phi=\text{undetermined})$$

$Y_l^m(\theta=0)$ 은 $m\neq 0$ 에서는 0.

$$\langle l\, m | \hat{z}\rangle = Y_l^{m*}(\theta=0, \phi=\text{undetermined})\, \delta_{m0}$$
$$= Y_l^{0*}\Big|_{\theta=0} \delta_{m0}$$
$$= \sqrt{\frac{2l+1}{4\pi}} P_l(\cos 0)\, \delta_{m0}$$
$$= \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0}$$

$$\boxed{Y_l^0 = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)}$$

$$\langle l, m' | \hat{n}\rangle = \sum_m D^{l}_{m'm}(\alpha=\phi, \beta=\theta, \gamma=0) \sqrt{\frac{2l+1}{4\pi}}\, \delta_{m0} = \sqrt{\frac{2l+1}{4\pi}} D^{l}_{m'0}(\alpha=\phi, \beta=\theta, \gamma=0)$$

$$D^{l}_{m'0}(\alpha=\phi, \beta=\theta, \gamma=0) = \sqrt{\frac{4\pi}{2l+1}} Y_l^{m'*}(\phi, \theta)$$

$$D^{(l)}_{m'0}(\alpha, \beta, \gamma) = e^{-im'\alpha} d^{(l)}_{m'0}(\beta)$$

$\alpha = \phi = 0$ 대입.

$$d^{(l)}_{m'0}(\beta) = \sqrt{\frac{4\pi}{2l+1}} Y_l^{m'*}(\theta=\beta, \phi=0)$$

$$d^{(2)}_{m'0}(\beta) = \sqrt{\frac{4\pi}{5}}\; Y_2^{m'*}(\theta=\beta, \phi=0)$$

$$\boxed{\begin{aligned} \text{Appendix} \quad Y_2^0 &= \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1) \\ Y_2^{\pm1} &= \mp\sqrt{\frac{15}{8\pi}}\,(\sin\theta\cos\theta)\, e^{\pm i\phi} \\ Y_2^{\pm2} &= \sqrt{\frac{15}{32\pi}}\,(\sin^2\theta)\, e^{\pm 2i\phi} \end{aligned}}$$

$$Y_2^{0*}(\theta=\beta, \phi=0) = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\beta - 1)$$

$$\left(Y_2^{\pm1}\right)^*(\theta=\beta, \phi=0) = \mp\sqrt{\frac{15}{8\pi}}\,(\sin\beta\cos\beta)$$

$$\left(Y_2^{\pm2}\right)^*(\theta=\beta, \phi=0) = \sqrt{\frac{15}{32\pi}}\,(\sin^2\beta)$$

회전된 게 $m=0$로 발견될 확률: $\left|d^{(2)}_{00}(\beta)\right|^2 = \frac{4\pi}{5}\frac{5}{16\pi}(3\cos^2\beta-1)^2 = \frac{1}{4}(3\cos^2\beta-1)^2$

〃 $m=\pm1$로 발견될확률: $\left|d^{(2)}_{\pm10}(\beta)\right|^2 = \frac{4\pi}{5}\frac{15}{8\pi}(\sin\beta\cos\beta)^2 = \frac{3}{2}(\sin\beta\cos\beta)^2$

〃 $m=\pm2$로 발견될 확률: $\left|d^{(2)}_{\pm20}(\beta)\right|^2 = \frac{4\pi}{5}\frac{15}{32\pi}(\sin^2\beta)^2 = \frac{3}{8}\sin^4\beta$

검산 → 모든 확률의 합이 1이 맞나?

$\mathbb{C} = \cos^2\beta$, $1-\mathbb{C} = \sin^2\beta$, $(\$+\mathbb{C})^2$

$$\left|d^{(2)}_{00}(\beta)\right|^2 = \frac{1}{4}(3\cos^2\beta-1)^2 = \frac{1}{4}(3\mathbb{C}-1)^2 = \frac{1}{4}(9\mathbb{C}^2-6\mathbb{C}+1)$$

$$\left|d^{(2)}_{+10}(\beta)\right|^2 + \left|d^{(2)}_{-10}(\beta)\right|^2 = 3(\sin\beta\cos\beta)^2 = 3\mathbb{C}(1-\mathbb{C}) = \frac{1}{4}(-12\mathbb{C}^2+12\mathbb{C})$$

$$\left|d^{(2)}_{+20}(\beta)\right|^2 + \left|d^{(2)}_{-20}(\beta)\right|^2 = \frac{3}{4}\sin^4\beta = \frac{3}{4}(1-\mathbb{C})^2 = \frac{1}{4}(3\mathbb{C}^2-6\mathbb{C}+3)$$

다 합하면 $\frac{4}{4}$, 1 나온다.

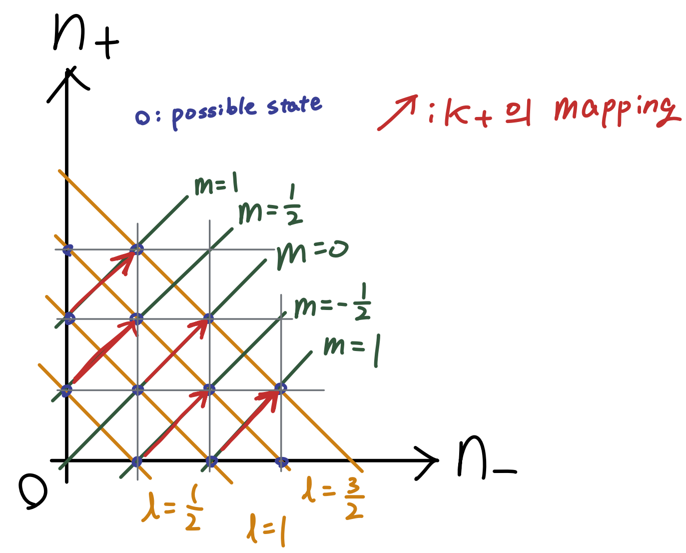
3.32 What is the physical significance of the operators

$$K_+ \equiv a_+^\dagger a_-^\dagger \quad \text{and} \quad K_- \equiv a_+ a_-$$ → m 값은 유지하고 j가 변화?

in Schwinger's scheme for angular momentum? Give the nonvanishing matrix elements of $K_\pm$.

물리적 의미:

K_+ 는 positive-type oscillator 를 하나 늘리고, negative-type oscillator 도 하나 늘린다.

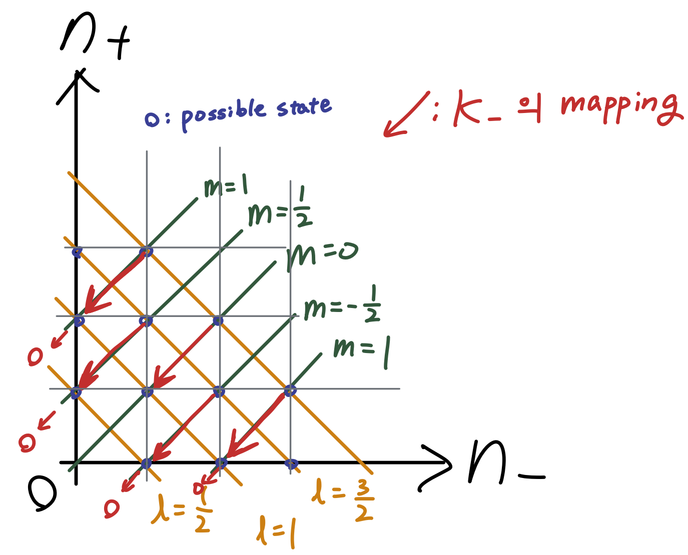


오른쪽 그림은 $j=\frac{1}{2}(n_+ + n_-)$, $m=\frac{1}{2}(n_+ - n_-)$ 을 이용해 states 를 나타낸 것이다.

$n_+ \longrightarrow n_+ + 1$, $n_- \longrightarrow n_- + 1$ 이면 결과적으로 $j \longrightarrow j+1$, $m \longrightarrow m$ 의 상태가 된다.

$$K_+ |j\ m\rangle = K^+_{jm} |j+1\ m\rangle$$

K^+_{jm} 은 계수이며, 구해야 하는 값이다.



K_- 는 K_+ 와 정확히 반대의 기능을 한다.

다만, $1-j \le m \le j-1$ 이어야 한다.

ladder operator 의 특성

$$\begin{cases} a_+ |n_+ = 0\ \ n_- = \text{undetermined}\rangle = 0 \\ a_- |n_+ = \text{undetermined}\ \ n_- = 0\rangle = 0 \end{cases}$$ 에 의해서,

$m = \pm j$ 인 경우 K_- 를 취하면 0 이 될 것이다.

$$K_- |j\ m\rangle = K^-_{jm} |j-1\ m\rangle$$

$$K_- |j\ j\rangle = 0$$

$$K_- |j\ -j\rangle = 0$$

계수 K 를 구해보자.

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$$\langle j\ m | K_+^\dagger K_+ | j\ m\rangle = |K_{jm}^+|^2 \langle j+1\ m | j+1\ m\rangle$$

$$\langle j\ m | K_+^\dagger K_+ | j\ m\rangle = \langle j\ m | K_- K_+ | j\ m\rangle$$

$$= \langle j\ m | a_+ a_- a_+^\dagger a_-^\dagger | j\ m\rangle$$

다른 type의 ladder operator는 commute

$$= \langle j\ m | a_+ a_+^\dagger\ a_- a_-^\dagger | j\ m\rangle$$

$$= \langle j\ m | N_+^\dagger N_-^\dagger | j\ m\rangle$$

$$= (n_-(j,m)+1)\ (n_+(j,m)+1)$$

$$= (j-m+1)(j+m+1)$$

$$\therefore |K_{jm}^+|^2 = (j-m+1)(j+m+1)$$

$$\langle j\ m | K_-^\dagger K_- | j\ m\rangle = |K_{jm}^-|^2 \langle j-1\ m | j-1\ m\rangle$$

$$\langle j\ m | K_-^\dagger K_- | j\ m\rangle = \langle j\ m | a_-^\dagger a_+^\dagger a_- a_+ | j\ m\rangle$$

$$= \langle j\ m | a_-^\dagger a_- a_+^\dagger a_+ | j\ m\rangle$$

$$= \langle j\ m | N_- N_+ | j\ m\rangle$$

$$= n_-(j,m)\ n_+(j,m) \langle j\ m | j\ m\rangle$$

$$= (j-m)(j+m)$$

$$\therefore |K_{jm}^-|^2 = (j-m)(j+m)$$

matrix element 를 구하면,

$$\langle j'\ m' | K_+ | j\ m\rangle = K_{jm}^+\ \delta_{j',j+1}\ \delta_{m'm} = \sqrt{(j+m+1)(j-m+1)}\ \delta_{j',j+1}\ \delta_{m'm}$$

$$\langle j'\ m' | K_- | j\ m\rangle = K_{jm}^-\ \delta_{j',j-1}\ \delta_{m'm} = \sqrt{(j+m)(j-m)}\ \delta_{j',j-1}\ \delta_{m'm}$$