QM 14th week Homework 김신지 20262182

3.35 A spin $\frac{1}{2}$ particle is in an orbital angular momentum $l=1$ state.

a. Starting with the total angular momentum state $|j,m\rangle = |j,j\rangle$ for $j=3/2$, use the operator $\mathbf{J}=\mathbf{L}+\mathbf{S}$ to construct all the states $|j,m_j\rangle$ in terms of the eigenstates $|l,m_l\rangle$ for $\mathbf{L}^2$ and L_z, and the eigenstates $|1/2,\pm 1/2\rangle$ for $\mathbf{S}^2$ and S_z. Check your answers against any available table of Clebsch–Gordan coefficients; see Appendix E.

b. If the particle is in a total angular-momentum eigenstate with z-component $+\hbar/2$, calculate the probability of finding the z-component of the spin of the particle to have the value $m_s=+1/2$.

Appendix

44. Clebsch-Gordan Coefficients, Spherical Harmonics, and *d* Functions

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J M	J M	...
m_1	m_2	Coefficients		
m_1	m_2			
⋮	⋮			

1/2×1/2

m_1 m_2	1 +1	1 0	0 0	1 −1
+1/2 +1/2	1			
+1/2 −1/2		1/2	1/2	
−1/2 +1/2		1/2	−1/2	
−1/2 −1/2				1

1×1/2

m_1 m_2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

2×1

	3 +3	3	2

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

	5/2 +5/2	5/2	3/2

2×1/2

m_1 m_2	5/2 +5/2	5/2 +3/2	3/2 +3/2	5/2 +1/2	3/2 +1/2	5/2 −1/2	3/2 −1/2	5/2 −3/2	3/2 −3/2	5/2 −5/2
+2 +1/2	1									
+2 −1/2		1/5	4/5							
+1 +1/2		4/5	−1/5							
+1 −1/2				2/5	3/5					
0 +1/2				3/5	−2/5					
0 −1/2						3/5	2/5			
−1 +1/2						2/5	−3/5			
−1 −1/2								4/5	1/5	
−2 +1/2								1/5	−4/5	
−2 −1/2										1

3/2×1/2

m_1 m_2	2 +2	2 +1	1 +1	2 0	1 0	2	1
+3/2 +1/2	1						
+3/2 −1/2		1/4	3/4				
+1/2 +1/2		3/4	−1/4				
+1/2 −1/2				1/2	1/2		

a. 풀이

L_z 에 대한 eigen value 가 최대 $\left(m=\frac{3}{2}\right)$ 인 상태는

$\{J^2, J_z\}$ basis 와 $\{\mathbb{L}^2, L_z, \mathbb{S}^2, S_z\}$ basis 모두에서 eigen ket 이다. 따라서,

$$\left|j=\tfrac{3}{2}, m=\tfrac{3}{2}\right\rangle = \left|m_l=1, m_s=\tfrac{1}{2}\right\rangle$$

이 상태에서 lowering operator $J_- = L_- + S_-$ 를 적용.

$$J_-\left|j=\tfrac{3}{2}, m=\tfrac{3}{2}\right\rangle = \sqrt{\left(\tfrac{3}{2}+\tfrac{3}{2}\right)(+1)}\left|j=\tfrac{3}{2}, m=\tfrac{1}{2}\right\rangle = \sqrt{3}\left|j=\tfrac{3}{2}, m=\tfrac{1}{2}\right\rangle$$

$$(L_- + S_-)\left|m_l=1, m_s=\tfrac{1}{2}\right\rangle = \sqrt{2}\left|m_l=0, m_s=\tfrac{1}{2}\right\rangle + \left|m_l=1, m_s=-\tfrac{1}{2}\right\rangle$$

$$\left|j=\frac{3}{2}, m=\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|m_l=0, m_s=\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|m_l=1, m_s=-\frac{1}{2}\right\rangle$$

한 번 더 양 변에 $J_- = L_- + S_-$ 를 적용.

$$2\left|j=\frac{3}{2}, m=-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left\{\sqrt{2}\left|m_l=-1, m_s=\frac{1}{2}\right\rangle + \left|m_l=0, m_s=-\frac{1}{2}\right\rangle\right.$$
$$+ \frac{1}{\sqrt{3}}\cdot\sqrt{2}\left|m_l=0, m_s=-\frac{1}{2}\right\rangle$$

$$= \frac{2}{\sqrt{3}}\left|m_l=-1, m_s=\frac{1}{2}\right\rangle + 2\cdot\sqrt{\frac{2}{3}}\left|m_l=0, m_s=-\frac{1}{2}\right\rangle$$

$$\left|j=\frac{3}{2}, m=-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|m_l=-1, m_s=\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|m_l=0, m_s=-\frac{1}{2}\right\rangle$$

다시 적용. ... 하기엔, 마지막 결과는 뻔하다.

$$\left|j=\frac{3}{2}, m=-\frac{3}{2}\right\rangle = \left|m_l=-1, m_s=-\frac{1}{2}\right\rangle$$

우리가 아직 모르는 것은,

$$\begin{cases} \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle = a\left|m_l=1, m_s=-\frac{1}{2}\right\rangle + b\left|m_l=0, m_s=\frac{1}{2}\right\rangle \\ \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle = c\left|m_l=0, m_s=-\frac{1}{2}\right\rangle + d\left|m_l=-1, m_s=\frac{1}{2}\right\rangle \end{cases}$$

이 관계에서 계수 a, b, c, d 를 모른다. 이건 그냥 orthogonality 이용해서 구할 거다.

이미 구한 거

$$\begin{cases} \left|j=\frac{3}{2}, m=\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|m_l=0, m_s=\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|m_l=1, m_s=-\frac{1}{2}\right\rangle \\ \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle = a\left|m_l=1, m_s=-\frac{1}{2}\right\rangle + b\left|m_l=0, m_s=\frac{1}{2}\right\rangle \end{cases}$$

$$\left\langle j=\frac{1}{2}, m=\frac{1}{2}\middle| j=\frac{3}{2}, m=\frac{1}{2}\right\rangle = \frac{a}{\sqrt{3}} + b\sqrt{\frac{2}{3}} = 0$$

$a = -\sqrt{2}\,b.$

이 정보만으로는 a와 b의 부호를 결정할 수 없다.

convention에 따라, m_l이 큰 $\left|m_l=1, m_s=-\frac{1}{2}\right\rangle$의 coefficient를 양수로 잡는다.

$|a|^2+|b|^2=1$ normalization 을 진행하면,

$a=\sqrt{\frac{2}{3}}$ $\quad b=-\frac{1}{\sqrt{3}}$

$$\begin{cases} \left|j=\frac{3}{2}, m=-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|m_l=-1, m_s=\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|m_l=0, m_s=-\frac{1}{2}\right\rangle \\ \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle = c\left|m_l=0, m_s=-\frac{1}{2}\right\rangle + d\left|m_l=-1, m_s=\frac{1}{2}\right\rangle \end{cases}$$

이미 구한거.

$$\left\langle j=\frac{1}{2}, m=-\frac{1}{2} \middle| j=\frac{3}{2}, m=-\frac{1}{2}\right\rangle = c\sqrt{\frac{2}{3}} + d\frac{1}{\sqrt{3}} = 0$$

$d=-\sqrt{2}\,c$, 역시 convention 에 따라 $c>0$ 으로 잡고, normalization,

$$c=\frac{1}{\sqrt{3}}, \quad d=-\sqrt{\frac{2}{3}}$$

결과를 모두 모아 보면,

$1\times1/2$

m_l m_s	$j=3/2$, $m=+3/2$
+1 +1/2	1

m_l m_s	$j=3/2$, $m=+1/2$	$j=1/2$, $m=+1/2$
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_l m_s	$j=3/2$, $m=-1/2$	$j=1/2$, $m=-1/2$
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_l m_s	$j=3/2$, $m=-3/2$
−1 −1/2	1

2×1 3 +3 3 2

$$\begin{cases} \left|j=\frac{3}{2}, m=\frac{3}{2}\right\rangle = \left|m_l=1, m_s=\frac{1}{2}\right\rangle \\ \left|j=\frac{3}{2}, m=\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|m_l=1, m_s=-\frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|m_l=0, m_s=\frac{1}{2}\right\rangle \\ \left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|m_l=1, m_s=-\frac{1}{2}\right\rangle - \frac{1}{\sqrt{3}}\left|m_l=0, m_s=\frac{1}{2}\right\rangle \\ \left|j=\frac{3}{2}, m=-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|m_l=0, m_s=-\frac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|m_l=-1, m_s=\frac{1}{2}\right\rangle \\ \left|j=\frac{1}{2}, m=-\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|m_l=0, m_s=-\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|m_l=-1, m_s=\frac{1}{2}\right\rangle \\ \left|j=\frac{3}{2}, m=-\frac{3}{2}\right\rangle = \left|m_l=-1, m_s=-\frac{1}{2}\right\rangle \end{cases}$$

결과가 기존에 알려진 CG와 일치한다!

b. 풀이 상태의 j 값이 명확하지 않으니 두 경우로 나누어 고려.

$\left|j=\frac{3}{2}, m=\frac{1}{2}\right\rangle$ 인 경우, $m_s=\frac{1}{2}$ 일 확률은 곧 $\left|\left\langle m_l=0\ m_s=\frac{1}{2} \middle| j=\frac{3}{2}, m=\frac{1}{2}\right\rangle\right|^2 = \frac{2}{3}$

$\left|j=\frac{1}{2}, m=\frac{1}{2}\right\rangle$ 인 경우, $m_s=\frac{1}{2}$ 일 확률은 곧 $\left|\left\langle m_l=0, m_s=\frac{1}{2} \middle| j=\frac{1}{2}, m=\frac{1}{2}\right\rangle\right|^2 = \frac{1}{3}$

3.42 Consider a spherical tensor of rank 1 (that is, a vector)

$$V_{\pm 1}^{(1)} = \mp \frac{V_x \pm i V_y}{\sqrt{2}}, \quad V_0^{(1)} = V_z.$$

Using the expression for $d^{(j=1)}$ given in Problem 3.38, evaluate

$$\sum_{q'} d_{qq'}^{(1)}(\beta) V_{q'}^{(1)}$$

and show that your results are just what you expect from the transformation properties of $V_{x,y,z}$ under rotations about the y-axis.

problem 3.38

$$d^{(j=1)}(\beta) = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & \left(-\frac{1}{\sqrt{2}}\right)\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

$$\sum_{q'} d_{qq'}^{(1)} V_{q'}^{(1)} = \begin{pmatrix} \frac{1}{2}(1+\cos\beta) & \left(-\frac{1}{\sqrt{2}}\right)\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix} \begin{pmatrix} \frac{-V_x - iV_y}{\sqrt{2}} \\ V_z \\ \frac{V_x - iV_y}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}\left(-\cos\beta V_x - i V_y - \sin\beta V_z\right) \\ -\sin\beta V_x + \cos\beta V_z \\ \frac{1}{\sqrt{2}}\left(\cos\beta V_x - i V_y + \sin\beta V_z\right) \end{pmatrix} = \begin{pmatrix} \frac{-V_x' - iV_y'}{\sqrt{2}} \\ V_z' \\ \frac{V_x' - iV_y'}{\sqrt{2}} \end{pmatrix}$$

$$R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}$$

$$\begin{pmatrix} V_x' \\ V_y' \\ V_z \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \cos\beta V_x + \sin\beta V_z \\ V_y \\ -\sin\beta V_x + \cos V_z \end{pmatrix}$$

김신지 20262182

$$\begin{cases} V_x' + iV_y' = \cos\beta V_x + \sin\beta V_z + iV_y \\ V_z' = -\sin\beta V_x + \cos\beta V_z \\ V_x' - iV_y' = \cos\beta V_x + \sin\beta V_z - iV_y \end{cases}$$

$V_{x,y,z}$ 가 다음 변환식을 따른다는 걸 알수있다.

$$\begin{cases} V_x' = \cos\beta V_x + \sin\beta V_z \\ V_y' = V_y \\ V_z' = -\sin\beta V_x + \cos\beta V_z \end{cases}$$