QM HW2.

P. 1.2

증명하라.

$$[AB, CD] = -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB$$

풀이

$$
\begin{aligned}
[AB, CD] &= ABCD - CDAB \\
&= [ABCD + ACBD] - [CDAB + CADB] - ACBD + CADB \\
&= A\{B,C\}D - C\{A,D\}B - [ACBD + ACDB] + [CADB + ACDB] \\
&= A\{B,C\}D - C\{A,D\}B - AC\{B,D\} + \{A,C\}DB \\
&= -AC\{D,B\} + A\{C,B\}D - C\{D,A\}B + \{C,A\}DB.
\end{aligned}
$$

P. 1.7 (a) 두 개의 ket $|\alpha\rangle$ 과 $|\beta\rangle$. complete한 base kets $|a'\rangle, |a''\rangle \ldots$ 들이 있을 때, $\langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \cdots$ 과 $\langle a'|\beta\rangle, \langle a''|\beta\rangle \cdots$ 가 알려져 있다. 이 basis 에서 $|\alpha\rangle\langle\beta|$ 의 matrix representation 을 찾아라.

풀이

$|\alpha\rangle$ 와 $|\beta\rangle$ 를 column vector 로 나타내면, $|a'\rangle, |a''\rangle \ldots$ basis 를 사용할 때,

$$|\alpha\rangle = \begin{pmatrix} \langle a'|\alpha\rangle \\ \langle a''|\alpha\rangle \\ \vdots \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} \langle a'|\beta\rangle \\ \langle a''|\beta\rangle \\ \vdots \end{pmatrix}$$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} \langle a'|\alpha\rangle \\ \langle a''|\alpha\rangle \\ \vdots \end{pmatrix} \begin{pmatrix} \langle a'|\beta\rangle^* & \langle a''|\beta\rangle^* & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} \langle a'|\alpha\rangle\langle a'|\beta\rangle^* & \langle a'|\alpha\rangle\langle a''|\beta\rangle^* & \cdots \\ \langle a''|\alpha\rangle\langle a'|\beta\rangle^* & \langle a''|\alpha\rangle\langle a''|\beta\rangle^* & \\ \vdots & & \ddots \end{pmatrix}$$

QM HW1.

P. 1.7 (b) spin $\frac{1}{2}$ system 을 생각하라. $|\alpha\rangle = |S_z;+\rangle$ $|\beta\rangle = |S_z;-\rangle$

이 basis로 $|\alpha\rangle\langle\beta|$ 의 matrix를 정확히 표현하라.

풀이.

$$|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\alpha\rangle\langle\beta| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

P. 1.10 $|+\rangle$ 과 $|-\rangle$ 의 orthonormality 를 이용하여 다음을 증명하라.

$$[S_i, S_j] = i\,\varepsilon_{ijk}\hbar S_k \qquad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$$

$$S_x \equiv \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \qquad S_y \equiv \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)$$

$$S_z \equiv \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)$$

풀이

먼저 $[S_i, S_j] = i\,\varepsilon_{ijk}\hbar S_k$ 부터 증명. S_xS_y, S_yS_x, S_yS_z, S_zS_y, S_zS_x, S_xS_z 를 직접 구해 보겠다.

$$S_xS_y = i\frac{\hbar^2}{4}\left(-\cancel{|+\rangle\langle-|+\rangle\langle-|} + |+\rangle\langle-|-\rangle\langle+| - |-\rangle\langle+|+\rangle\langle-| + \cancel{|-\rangle\langle+|-\rangle\langle+|}\right)$$

$$= i\frac{\hbar^2}{4}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) = i\frac{\hbar}{2}S_z$$

$$S_yS_x = i\frac{\hbar^2}{4}\left(-\cancel{|+\rangle\langle-|+\rangle\langle-|} - |+\rangle\langle-|-\rangle\langle+| + |-\rangle\langle+|+\rangle\langle-| + \cancel{|-\rangle\langle+|-\rangle\langle+|}\right)$$

$$= i\frac{\hbar^2}{4}\left(-|+\rangle\langle+| + |-\rangle\langle-|\right) = -i\frac{\hbar}{2}S_z$$

$$S_yS_z = i\frac{\hbar^2}{4}\left(-\cancel{|+\rangle\langle-|+\rangle\langle+|} + |+\rangle\langle-|-\rangle\langle-| + |-\rangle\langle+|+\rangle\langle+| - \cancel{|-\rangle\langle+|-\rangle\langle-|}\right)$$

$$= i\frac{\hbar^2}{4}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) = i\frac{\hbar}{2}S_x$$

$$S_zS_y = i\frac{\hbar^2}{4}\left(-|+\rangle\langle+|+\rangle\langle-| + \cancel{|+\rangle\langle+|-\rangle\langle+|} + \cancel{|-\rangle\langle-|+\rangle\langle-|} - |-\rangle\langle-|-\rangle\langle+|\right)$$

$$= i\frac{\hbar^2}{4}\left(-|+\rangle\langle-| - |-\rangle\langle+|\right) = -i\frac{\hbar}{2}S_x$$

$$S_zS_x = \frac{\hbar^2}{4}\left(|+\rangle\langle+|+\rangle\langle-| + \cancel{|+\rangle\langle-|+\rangle\langle+|} - \cancel{|-\rangle\langle-|+\rangle\langle-|} - |-\rangle\langle-|-\rangle\langle+|\right)$$

$$= \frac{\hbar^2}{4}\left(|+\rangle\langle-| - |-\rangle\langle+|\right) = i\frac{\hbar}{2}S_y$$

$$S_x S_z = \frac{\hbar^2}{4}\left(\cancel{|+\rangle\langle -|+\rangle\langle +|} - |+\rangle\langle -|-\rangle\langle -| + |-\rangle\langle +|+\rangle\langle +| - \cancel{|-\rangle\langle +|-\rangle\langle -|} \right)$$

$$= \frac{\hbar^2}{4}\left(-|+\rangle\langle -| + |-\rangle\langle +| \right) = -i\frac{\hbar}{2} S_y$$

$[S_x, S_y] = S_x S_y - S_y S_x = i\frac{\hbar}{2} S_z + i\frac{\hbar}{2} S_z = i\hbar S_z$

$[S_y, S_z]$ 와 $[S_z, S_x]$ 도 똑같은 방식으로 증명되어 $[S_y, S_z] = i\hbar S_x$, $[S_z, S_x] = i\hbar S_y$

$[S_y, S_x]$, $[S_z, S_y]$, $[S_x, S_z]$ 는 commutator의 anti-symmetry로 증명된다. $[A, B] = -[B, A]$ 이기에

$[S_y, S_x] = -i\hbar S_z$, $[S_z, S_y] = -i\hbar S_x$, $[S_x, S_z] = -i\hbar S_y$.

만약 같은 S끼리 곱하면 어찌 되는가? S_x, S_y, S_z 모두 matrix로 나타내면 orthogonal matrix이다. 그래서 제곱하면 항등행렬 $\mathbb{1}$ 의 스칼라배가 나온다.

$$S_x S_x = S_y S_y = S_z S_z = \frac{\hbar^2}{4}\left(|+\rangle\langle +| + |-\rangle\langle -| \right) = \frac{\hbar^2}{4}\mathbb{1}$$

결론 :

$$\left\{ \begin{array}{l} S_x S_y = i\frac{\hbar}{2} S_z \\ S_y S_z = i\frac{\hbar}{2} S_x \\ S_z S_x = i\frac{\hbar}{2} S_y \\ S_y S_x = -i\frac{\hbar}{2} S_z \\ S_z S_y = -i\frac{\hbar}{2} S_x \\ S_x S_z = -i\frac{\hbar}{2} S_y \\ S_x S_x = S_y S_y = S_z S_z = \frac{\hbar^2}{4}\mathbb{1} \end{array} \right.$$

요약하면, $S_i S_j = i\frac{\hbar}{2}\varepsilon_{ijk} S_k + \frac{\hbar^2}{4}\mathbb{1}\delta_{ij}$, 레비치비타 반대칭성 $\varepsilon_{ijk} = -\varepsilon_{jik}$ 이용

$$[S_i, S_j] = S_i S_j - S_j S_i = i\frac{\hbar}{2}\varepsilon_{ijk} S_k - i\frac{\hbar}{2}\varepsilon_{jik} S_k + \frac{\hbar^2}{4}\mathbb{1}(\delta_{ij} - \delta_{ji})$$

$$= i\frac{\hbar}{2}\varepsilon_{ijk} S_k + i\frac{\hbar}{2}\varepsilon_{ijk} S_k$$

$$= i\hbar\,\varepsilon_{ijk} S_k.$$

$$\{S_i, S_j\} = S_i S_j + S_j S_i = \left(i\frac{\hbar}{2}\varepsilon_{ijk} + i\frac{\hbar}{2}\varepsilon_{jik}\right) S_k + \frac{\hbar^2}{2}\mathbb{1}\delta_{ij} = \frac{\hbar^2}{2}\mathbb{1}\delta_{ij}$$